Notes for Math 760 – Topics in Differential Geometry

January 2015

0 Introduction: Problems in Riemannian Geometry

Undefined terms: Differentiable manifold, Riemannian manifold, homogeneous manifold, incompressible submanifold, stabilizer, conformal class

Obviously differential geometry is a sub-study of differentiable manifold theory, and while geometry has its own internal motivations and problems, to a substantial degree geometry exists to serve topology. In some sense the central project of differential topology is to establish a set of numerical or algebraic invariants that can perfectly distinguish manifolds from one another, as the algebraic invariant $a^2 + b^2 = c^2$ distinguishes right triangles in Euclidean space from all other triangles (or, more pedantically, the numerical invariants a, b, c (side lengths) distinguish all triangles up to similarity).

Nineteenth century mathematics determined that compact 2-manifolds could be classified by a single numerical invariant: the Euler number (and when $\chi = 0$ also orientability). Part of the cluster of results surrounding this was the *geometrization* (also known as uniformization) of compact 2-manifolds: any compact 2-manifold possesses a metric of constant curvature +1, 0, or -1.

Later, the ideas of geometrization leached into other areas of manifold theory; of special note is the case of 3-manifolds. As opposed to the 3 homogeneous geometries of 2-manifolds, Thurston was noticed that 3-manifolds could admit just 8 homogeneous geometries (with compact stabilizers), called the *model geometries*. He noticed that, although incompressible spheres and tori obstructed the existence of a model geometry, incompressible 2-manifolds of other kinds did not. This, along with other evidence, led Thurston conjecture that any 3-manifold can be cut along incompressible tori (in an essentially unique way) such that the resulting connected components admit a complete metric with one of the 8 geometries.

Since the topological structure of any manifold with a model geometry is completely understood, if any compact 3-manifold admits such a decomposition, then the topology of all compact 3-manifolds would be understood.

Starting in the early 1980's, Richard Hamilton began a program of using the Ricci Flow to prove the geometrization conjecture for 3-manifolds. The Ricci flow is an evolution equation for the metric:

$$\frac{dg_{ij}}{dt} = -2\mathrm{Ric}_{ij}. \tag{1}$$

This is a non-linear 2nd order PDE, which resembles a heat equation, or more accurately a reaction-diffusion equation. Hamilton proved that if singularities do not develop, the resulting manifold in the limit carries a model geometry. Singularities proved difficult to control, but in 2003 Perelman classified the singularities, proved that there are at most finitely many of them, and thereby completed the geometrization program.

The great success of differential-geometric methods in the solution of Thurston's ostensibly topological program has spurred development of such methods in other dimensions. To simplify (almost to absurdity!), the program has three parts

1) In dimension n, determine the "canonical" geometries, and the topological restrictions associated to such a geometry
2) Determine a canonical way to break a manifold into pieces, each of which can be given a canonical geometry
3) Determine the topology of the original manifold from the topology of the pieces with their gluing pattern

By-and-large, we the mathematical community is stuck on (1), and we expect to be stuck there for the rest of our lives! (Except in some important special cases.) Surely Einstein metrics should be canonical; then as an example of the difficulty of (1), you may ask *which manifolds admit an Einstein metric?* If you answer this, even in the case $n = 4$, you'll get a fields medal. However we now completely understand which *Kähler* manifolds admit Kähler-Einstein metrics.

Many of the problems we'll explore are involved in the search for canonical metrics. Our list of topics (we'll surely not get to all of them) is

- The Yamabe Problem: does a conformal class contain a constant-scalr curvature metric? Is it unique?
- (prescribed scalar curvature) Given a function $f : M^n \to \mathbb{R}$, is there a metric on M^n so that $scal = f$?
- The Ricci flow: can we canonically smooth-out a metric using a heat-like flow in curvature, like the heat flow "smooths out" any distribution of heat on a compact body? (Possibly related topic: the Yamabe flow.)
- Complex geometry, especially Kähler geometry: if we add additional structure, do our problems become more solvable? (Answer: Yes, but they're still hard.)

- The Calabi conjectures, and Yau's and Aubin's partial solutions (possible related topic: the Calabi flow)
- The Yang-Mills equation and the Yang-Mills flow: vector bundles are of decisive significance in Riemannian geometry. Do bundles have canonical geometries? Can we establish a heat-type flow to attempt to find them? (Yes and yes.)

1 Topic 1 – Bridging Analysis and Geometry

1.1 Stokes' Theorem

The primary nexus of analysis and differential topology is Stokes Theorem: if ω is a k-form on M^n and $N^{k+1} \subset M^n$ is a submanifold, then

$$\int_N d\omega \;=\; \int_{\partial N} \omega. \tag{2}$$

Stokes theorem is FTCII plus diffeomorphism invariance of integration.

1.2 The Sobolev Inequalities

The primary nexus of analysis and differential geometry is the cluster of results including and surrounding the *Sobolev embedding theorems*. These are familiar in the Euclidean setting, where they are usually proven (and understood) as consequences of the Poincare inequality, which is normally proven via iterated applications of FTCII (which is Stokes Theorem in dimension 1). But careful study of the proof shows that it is dependent on the Euclidean structure of $\mathbb{R}^n$; the proof cannot be imitated on manifolds. What's more, the statement of the Poincare inequality is very distinctly not diffeomorphism invariant.

The proof of the Sobolev inequalities of Federer-Fleming, dating to the late 1950's, showed us their fundamental connections with geometry; it is their proof that we'll give here. Before giving the proof, we introduce two constants. If (Ω, g) is a domain with metric g, then the ν-isometric inequality is

$$I_\nu(\Omega) \;=\; \inf_{\Omega_o \subset\subset \Omega} \; \frac{|\partial\Omega_o|}{|\Omega_o|^{\frac{\nu-1}{\nu}}} \tag{3}$$

where $|\partial\Omega_o|$ indicates the $(n-1)$ Hausdorff measure of $\partial\Omega_o$ and $|\Omega_o|$ indicates the n-Hausdorff measure of Ω_o. The ν-Sobolev constant is

$$C_\nu(\Omega) \;=\; \inf_{\substack{f\in C^\infty(\Omega)\\ Supp(f)\subset\Omega}} \; \frac{\int |\nabla f|}{\left(\int f^{\frac{\nu}{\nu-1}}\right)^{\frac{\nu-1}{\nu}}} \tag{4}$$

Clearly both are less than ∞.

Theorem 1.1 (Federer-Fleming)

$$I_\nu(\Omega) \;=\; C_\nu(\Omega)$$

Proof. Step I: $I_\nu(\Omega) \;\geq\; C_\nu(\Omega)$

Given any domain $\Omega_o \subset\subset \Omega$, define the function

$$f_\epsilon(x) \;=\; max\left\{1 - \frac{1}{\epsilon}\,\mathrm{dist}(x, \Omega_0),\;\; 0\right\} \tag{5}$$

It is easy to prove that $\lim_{\epsilon\to 0}\int f_\epsilon^{\frac{\nu-1}{\nu}} \;=\; |\Omega|$. It is somewhat more complicated, but still not difficult to prove that $\lim_{\epsilon\to 0}\int \nabla f_\epsilon \;=\; |\Omega_o|$. Therefore

$$|\partial\Omega_o| \;=\; \lim_{\epsilon\to 0}\int |\nabla f_\epsilon| \;\geq\; C_\nu(\Omega)\left(\int f_\epsilon^{\frac{\nu}{\nu-1}}\right)^{\frac{\nu-1}{\nu}} \;\geq\; C_\nu(\Omega)\,|\Omega_o|^{\frac{\nu-1}{\nu}}. \tag{6}$$

Step II: $C_\nu(\Omega) \;\geq\; I_\nu(\Omega)$

This is the more interesting case. Assume f is smooth, with isolated critical points. On any non-critical point p, let dA indicate the area measure on the level-set of f passing through p. The co-area formula is

$$dVol_p = \frac{1}{|\nabla f|} df \wedge dA \tag{7}$$

Then using $\Omega_t = \{|f| \geq t\}$ to denote the superlevel sets, we have

$$\begin{aligned}\int |\nabla f|\, dVol \;&=\; \int df\wedge dA \;=\; \int_{\inf|f|}^{\sup|f|}\int_{\{f=0\}} dA\, df \;=\; \int_0^\infty |\partial\Omega_t|\, dt\\ &\geq\; I_\nu(\Omega)\int_0^\infty |\Omega_t|^{\frac{\nu-1}{\nu}}\end{aligned} \tag{8}$$

and using the "layer-cake" representation and then a change of variables (a lá multivariable calculus), we have

$$\begin{aligned}\int f^{\frac{\nu}{\nu-1}} \;&=\; \frac{\nu}{\nu-1}\int_\Omega\int_0^{|f(p)|} t^{\frac{1}{\nu-1}}\, dt\, dVol(p) \;=\; \frac{\nu}{\nu-1}\int_0^\infty\int_{\Omega_t} t^{\frac{1}{\nu-1}}\, dVol\, dt\\ &=\; \frac{\nu}{\nu-1}\int_0^\infty t^{\frac{1}{\nu-1}}|\Omega_t|\, dt\end{aligned} \tag{9}$$

Lastly we have to connect these. We have

Lemma 1.2 *If $g(t)$ is non-negative and decreasing, and if $s \geq 1$, then*

$$\left(s\int_0^\infty t^{s-1}g(t)\, dt\right)^{\frac{1}{s}} \;\leq\; \int_0^\infty g(t)^{\frac{1}{s}}\, dt \tag{10}$$

Proof. First taking a derivative

$$\begin{aligned} \frac{d}{dT}\left(s\int_0^T t^{s-1}g(t)\,dt\right)^{\frac{1}{s}} &= T^{s-1}g(T)\left(s\int_0^T t^{s-1}g(t)\,dt\right)^{\frac{1-s}{s}} \\ &\le T^{s-1}g(T)^{\frac{1}{s}}\left(s\int_0^T t^{s-1}\right)^{\frac{1-s}{s}} = g(T)^{\frac{1}{s}} \end{aligned} \tag{11}$$

and then taking an integral, we get

$$\left(s\int_0^\infty t^{s-1}g(t)\right)^{\frac{1}{s}} \le \int_0^\infty g(t)^{\frac{1}{s}}\,dt. \tag{12}$$

□

Applying this lemma with $s = \frac{\nu}{\nu-1}$ we get

$$\left(\int f^{\frac{\nu-1}{\nu}}\right)^{\frac{\nu}{\nu-1}} = \left(\frac{\nu}{\nu-1}\int_0^\infty t^{\frac{1}{\nu-1}}|\Omega_t|\,dt\right)^{\frac{\nu-1}{\nu}} \le \int_0^\infty |\Omega_t|^{\frac{\nu-1}{\nu}} \le \frac{1}{I_\nu(\Omega)}\int|\nabla f| \tag{13}$$

giving the inequality $I_\nu(\Omega) \le C_\nu(\Omega)$. □

Theorem 1.3 *For any $p < \nu$ we obtain embeddings $L^{\frac{p\nu}{\nu-p}} \subset W_0^{1,p}$.*

The critical value of ν is $\nu = n$. The Sobolev constant on standard Euclidean space has $C_n(\mathbb{R}^n)$ nonzero, but for any other value of ν, $C_\nu(\mathbb{R}^n) = 0$.

If f has more than one weak derivative, then the embedding theorem also improves.

Theorem 1.4 *If $1 \le p < \infty$, $k \in \mathbb{N}$, and $kp < \nu$ then we obtain embeddings $L^{\frac{p\nu}{\nu-pk}} \subset W_0^{k,p}$.*

So what happens if $pk \ge \nu$? As $pk \nearrow \nu$, then $\frac{p\nu}{\nu-pk} \nearrow \infty$, so perhaps the embedding is into L^∞. This naive supposition fails because the constant in the Sobolev inequality also degenerates (see the exercises).

However if $pk > n$, then we obtain strong consequences indeed.

Theorem 1.5 *If $l + \alpha = k - \frac{n}{p}$ and $C_n(\Omega) > 0$, then*

$$W_0^{k,p}(\Omega) \subset C_0^{l,\alpha}(\Omega) \tag{14}$$

Our last word is the Kondrachov-Rellich embedding theorems:

Theorem 1.6 *The embeddings above are all compact embeddings.*

The proof is not geometrically interesting, so we'll skip it.

As a final note, notice if $\Omega = M^n$ is compact, obviously $C_\nu(\Omega) = I_\nu(\Omega) = 0$ for all ν. This is not useful, so we make the following modification:

$$\begin{aligned} I_\nu(\Omega) &= \inf_{\substack{\Omega_o \subset\subset \Omega \\ |\Omega_o| \le \frac{1}{2}|\Omega|}} \frac{|\partial \Omega_o|}{|\Omega_o|^{\frac{\nu-1}{\nu}}} \\ C_\nu(\Omega) &= \inf_{\substack{f \in C^\infty(\Omega) \\ Supp(f) \subset \Omega \\ |\operatorname{supp}(f)| \le \frac{1}{2}|\Omega|}} \frac{\int |\nabla f|}{\left(\int f^{\frac{\nu}{\nu-1}}\right)^{\frac{\nu-1}{\nu}}} \end{aligned} \tag{15}$$

1.3 Suggested Problems for 1/21

1) Prove that, if $\nu \neq n$, then $I_\nu(\mathbb{R}^n) = 0$.
2) If Ω is any pre-compact domain in $\mathbb{R}^n$ and if $\nu < n$, show that $I_\nu(\Omega) = 0$. Harder: If $\nu > n$, show that $I_\nu(\Omega) \neq 0$.
3) Prove Theorem 1.3 by considering the function f^γ for some γ, and showing that for any $1 \le p < \nu$ we have $\left(\int |\nabla f|^p\right)^{\frac{1}{p}} \ge C_\nu(\Omega) \frac{\nu-p}{p(\nu-1)} \left(\int f^{\frac{p\nu}{\nu-p}}\right)^{\frac{\nu-p}{p\nu}}$.
4) Prove Theorem 1.4 by iterating Theorem 1.3.

1.4 Laplacians

1.4.1 Covariant Derivatives and the Rough Laplacian

If T is a tensor, say

$$\begin{aligned} &T : \bigotimes{}^p TM \otimes \bigotimes{}^q T^*M \;\to\; \mathbb{R} \\ &T(X, X_{i_1}, \ldots, X_{i_p}, \eta^{j_1}, \ldots, \eta^{j_q}) \;\in\; \mathbb{R} \\ &T = T_{i_1 \ldots i_p}{}^{j_1 \ldots j_q} \;\; dx^{i_1} \otimes \cdots \otimes dx^{i_p} \otimes \frac{\partial}{\partial x^{j_i}} \otimes \cdots \otimes \frac{\partial}{\partial x^{j_q}} \end{aligned} \tag{16}$$

then we can define its covariant derivative, defined via a Leibniz rule:

$$\begin{aligned} (\nabla T)(X, X_{i_1}, \ldots, X_{i_p}, \eta^{j_1}, \ldots, \eta^{j_q}) \;&=\; X\left(T(X_{i_1}, \ldots, X_{i_p}, \eta^{j_1}, \ldots, \eta^{j_q})\right) \\ -\; T(\nabla_X X_{i_1}, \ldots, X_{i_p}, \eta^{j_1}, \ldots, \eta^{j_q}) \;&-\; \ldots \;-\; T(X_{i_1}, \ldots, X_{i_p}, \eta^{j_1}, \ldots, \nabla_X \eta^{j_q}) \end{aligned} \tag{17}$$

The tensor ∇T has an expression in coordinates, where, by convention, the derivative is indicated with a comma:

$$\nabla T \ = \ T_{i_1 \ldots i_p}{}^{j_1 \ldots j_q}{}_{,k} \ \ dx^{i_1} \otimes \ldots dx^{i_p} \otimes \frac{\partial}{\partial x^{j_1}} \otimes \cdots \otimes \frac{\partial}{\partial x^{j_1}} \otimes dx^k. \tag{18}$$

One can compute the following expression for the component functions:

$$\begin{aligned} T_{i_1 \ldots i_p}{}^{j_1 \ldots j_q}{}_{,k} \ &= \ \frac{d}{dx^k} T_{i_1 \ldots i_p}{}^{j_1 \ldots j_q} \ + \Gamma^s_{k i_1} T_{s \ldots i_p}{}^{j_1 \ldots j_q} \ + \ \ldots \ + \Gamma^s_{k i_p} T_{i_1 \ldots s}{}^{j_1 \ldots j_q} \\ &\quad + \Gamma^{j_1}_{ks} T_{i_1 \ldots s}{}^{s \ldots j_q} \ + \ \ldots \ + \Gamma^{j_p}_{ks} T_{i_1 \ldots s}{}^{j_1 \ldots s}. \end{aligned} \tag{19}$$

Obviously this can be iterated: ∇T, $\nabla^2 T$, $\nabla^3 T$ etc. The so-called *rough Laplacian* of a tensor T is the trace of the Hessian:

$$\triangle T \ = \ g^{kl} T_{i_1 \ldots i_p}{}^{j_1 \ldots j_q}{}_{,kl} \ \ dx^{i_1} \otimes \ldots dx^{i_p} \otimes \frac{\partial}{\partial x^{j_1}} \otimes \cdots \otimes \frac{\partial}{\partial x^{j_1}}. \tag{20}$$

In particular, if f is a function, then

$$\triangle f \ = \ g^{ij} \frac{\partial^2 f}{\partial x^i \partial x^j} \ - \ \Gamma^k \frac{df}{dx^k}, \qquad \Gamma^k \ \triangleq \ g^{ij} \Gamma^k_{ij}. \tag{21}$$

1.4.2 Forms, the Hodge Duality Operator, and the Hodge Laplacian

We recall a few definitions. We have the sections of k-forms, denoted $\Gamma\left(\bigwedge^k M^n\right)$. By abuse of notation, we denote this by simply $\bigwedge^k M^n$ or just $\bigwedge^k$. The *Hodge duality operator* or just *Hodge star* is term given to the tensoral forms of the volume form:

$$\begin{aligned} dVol \ &= \ \sqrt{g_{ij}}\, dx^1 \wedge \cdots \wedge dx^n, \\ * : \bigwedge{}^k \ &\longrightarrow \ \bigwedge{}^{n-k} \end{aligned} \tag{22}$$

The $*$ operator is an isomorphism, as $** = (-1)^{k(n-k)}$. It is also positive definite in the sense that $*\,(\eta \wedge *\eta) \geq 0$ with equality if and only if $\eta = 0$. Polarizing this quadratic operator, we define an inner product

$$\langle \eta, \, \gamma \rangle \ = \ *\,(\eta \wedge *\gamma)\,. \tag{23}$$

If $\eta, \gamma \in \bigwedge^k$ then $\eta \wedge *\gamma \in \bigwedge^n$, so we can integrate it. This gives rise to an L^2 inner product

$$(\eta, \, \gamma) \ \triangleq \ \int \eta \wedge *\gamma. \tag{24}$$

Now let's look at the exterior derivative $d : \bigwedge^k \to \bigwedge^{k+1}$. We would like to find its L^2 adjoint. Now d is a linear operator, but is only densely defined. If we restrict the L^2 space to just the C^∞ k-forms, then d is defined everywhere, but it is not a bounded (or continuous) linear operator, and the inner peoduct space is not a Hilbert space (due to non-completeness of C^∞ under the L^2 norm). Despite all this, we *can* find an adjoint for d, by making use of Stoke's theorem. We find that d^*, defined implicitly by

$$(d\eta,\, \gamma) \;=\; (\eta,\, d^*\gamma)\,. \tag{25}$$

One use of Stokes theorem along with the fact that $** = (-1)^{k(n-k)}$ gives us $d^* : \bigwedge^k \to \bigwedge^{k-1}$ is $d^*\eta = (-1)^{nk+k+1} * (d\,(*\eta))$.

The first order operators d, d^* can be combined give a second order operator, the *Hodge Laplacian*, by

$$\begin{aligned} \triangle_H \;&:\; \bigwedge{}^k \longrightarrow \bigwedge{}^k, \\ \triangle_H \eta \;&=\; d^* d\eta \,+\, dd^*\eta. \end{aligned} \tag{26}$$

This is not the same as the rough Laplacian, although at the first and second order levels they are essentially the same. Notice that $\triangle$ is negative definite and $\triangle_H$ is positive definite:

$$\begin{aligned} (\triangle\eta,\, \eta) \;&=\; -|\nabla\eta|^2 \\ (\triangle_H\eta,\, \eta) \;&=\; |d\eta|^2 \,+\, |d^*\eta|^2. \end{aligned} \tag{27}$$

It is possible to prove that we have *Bochner formulas* for the Laplacians

$$\triangle_H \eta \;=\; -\triangle\eta \,+\, F(g,\, \eta) \tag{28}$$

where F is a zero-order differential operator (a linear operator) in η, and a second-order non-linear operator in g. The following commutation relations are also often very useful:

$$\begin{aligned} [d, \triangle_H] \;&=\; d\triangle_H \,-\, \triangle_H d \;=\; 0 \\ [d^*, \triangle_H] \;&=\; d^*\triangle_H \,-\, \triangle_H d^* \;=\; 0 \\ [*, \triangle_H] \;&=\; *\triangle_H \,-\, \triangle_H * \;=\; 0. \end{aligned} \tag{29}$$

There are no such simple commutation relation for the rough Laplacian, although later we shall talk about the important commutator $[\triangle, \nabla]$.

1.5 Suggested Problems for 1/26

5) Prove formulae (19) and (21).

6) If f is a function, prove the classic formula $\triangle f \;=\; \frac{1}{\sqrt{\det g}} \frac{\partial}{\partial x^i} \left(\sqrt{\det g}\, g^{ij} \frac{\partial f}{\partial x^j} \right)$.

7) Prove $** : \bigwedge^k \to \bigwedge^k$ is the operator $(-1)^{k(n-k)}$.

8) Prove the codifferential operator is $d^* = (-1)^{nk+k+1} * d* : \bigwedge^k \to \bigwedge^{k-1}$.

9) Prove that the inner product $\langle \eta, \gamma \rangle$ is symmetric (hint: work on a basis).

10) Prove that a rough-harmonic form is constant, while a Hodge-harmonic form need not be constant, but must be both closed and co-closed.

11) In addition to the inner product (24), there is another inner product

$$\langle \eta, \gamma \rangle = g^{i_1 j_1} \dots g^{i_k j_k} \eta_{i_1 \dots i_k} \gamma_{j_1 \dots j_k}.$$

Show that the two are proportional, and find the constant of proportionality.

1.6 Sobolev Constants in Action: The Moser Iteration Method

Let $\eta : M^n \to \mathbb{R}^{\geq 0}$ be a *test function*, meaning $\eta \in C_c^\infty(M^n)$, and let f be any weakly differentiable function that weakly satisfies

$$\triangle f + uf \geq 0 \tag{30}$$

(more general elliptic equations can be considered, but for simplicity we'll stick to (30) as the method in the general case is essentially the same.) A standing assumption will be $f \geq 0$.

1.6.1 Initial Apriori Estimates

In addition to (30) assume $f \in W^{1,p}_{loc}$ for some $p > 1$ and assume $u \in L^{\frac{n}{2}}_{loc}$. Then an application of the divergence theorem along with the Cauchy-Schwartz inequality (and some propitious re-arrangement) gives

$$\begin{aligned}
(p-1)\int \eta^2 f^{p-2}|\nabla f|^2 &= \int \eta^2 \left\langle \nabla f^{p-1}, \nabla f \right\rangle \\
&= \int Div\left(\eta^2 \nabla f^{p-1} \nabla f\right) \\
&\quad - \int \left\langle \sqrt{p-1}\,\eta f^{\frac{p}{2}-1}\nabla f,\, \frac{2}{\sqrt{p-1}}\, f^{\frac{p}{2}}\nabla \eta \right\rangle - \int \eta^2 f^{p-2} \triangle f \\
&\leq \frac{p-1}{2}\int \eta^2 f^{p-2}|\nabla f|^2 + \frac{2}{p-1}\int |\nabla \eta|^2 f^p + \int \eta^2 f^p u
\end{aligned} \tag{31}$$

which gives us an equation that could be regarded as a form of reverse Sobolev inequality:

$$\int \eta^2 f^{p-2}|\nabla f|^2 \leq \left(\frac{2}{p-1}\right)^2 \int |\nabla \eta|^2 f^p + \frac{2}{p-1}\int \eta^2 f^p u \tag{32}$$

Set $\gamma = \frac{n}{n-2}$ and notice $\gamma > 1$ (this is convenient for applications of Hölder's inequality as $\frac{1}{\gamma} + \frac{2}{n} = 1$). One form of the Sobolev inequality is

$$C_n(\Omega)\left(\int g^{2\gamma}\right)^{\frac{1}{2\gamma}} \leq \left(\int |\nabla g|^2\right)^{\frac{1}{2}} \tag{33}$$

where $g \in W_0^{1,2}(\Omega)$. Using our "reverse Sobolev inequality" along with the actual Sobolev inequality gives

$$\begin{aligned} C_n(\mathrm{supp}\,\eta)^2\left(\int \eta^{2\gamma}f^{p\gamma}\right)^{\frac{1}{\gamma}} &\leq \int |\nabla(\eta f^{\frac{p}{2}})|^2 \\ &\leq 2\int|\nabla\eta|^2 f^p + \frac{p^2}{2}\int \eta^2 f^{p-2}|\nabla f|^2 \\ &\leq 2\left(1+\left(\frac{p}{p-1}\right)^2\right)\int|\nabla\eta|^2 f^p + \frac{p^2}{p-1}\int\eta^2 f^p u \end{aligned} \tag{34}$$

With Hölder's inequality we obtain

$$\begin{aligned} &C_n(\mathrm{supp}\,\eta)^2\left(\int \eta^{2\gamma}f^{p\gamma}\right)^{\frac{1}{\gamma}} \\ &\leq 2\left(1+\left(\frac{p}{p-1}\right)^2\right)\int|\nabla\eta|^2 f^p + \frac{p^2}{p-1}\left(\int\eta^{2\gamma}f^{p\gamma}\right)^{\frac{1}{\gamma}}\left(\int_{supp\,\eta}u^{\frac{n}{2}}\right)^{\frac{2}{n}} \end{aligned} \tag{35}$$

$$\left[C_n(\mathrm{supp}\,\eta)^2 - \frac{p^2}{p-1}\left(\int_{supp\,\eta}u^{\frac{n}{2}}\right)^{\frac{2}{n}}\right]\left(\int\eta^{2\gamma}f^{p\gamma}\right)^{\frac{1}{\gamma}} \leq 2\left(1+\left(\frac{p}{p-1}\right)^2\right)\int|\nabla\eta|^2 f^p$$

Thus, provided $\int_{\mathrm{supp}\ \eta} U^{\frac{n}{2}}$ is sufficiently small with respect to both p and the Sobolev constant, then the local $L^{\gamma p}$-norm of f is controlled in terms of the local L^p-norm of f. Lifting apriori from lower L^p-norms to higher L^p-norms is called *bootstrapping*.

Lemma 1.7 (First Bootstrapping Inequality) *Assume $\triangle f + uf \geq 0$, $f \geq 0$, $f \in L^p_{loc}$ for some $p > 1$, $u \in L^{\frac{n}{2}}_{loc}$, and $\eta \in C^\infty_c$. Then*

$$\int_{supp\,\eta} u^{\frac{n}{2}} \leq \frac{p-1}{2p^2}C_n(supp\,\eta)^2 \tag{36}$$

implies

$$\left(\int\eta^{2\gamma}f^{p\gamma}\right)^{\frac{1}{p\gamma}} \leq \left(\frac{4p^2}{p-1}\frac{1}{C_n(supp\,\eta)^2}\right)^{\frac{1}{p}}\left(\int|\nabla\eta|^2 f^p\right)^{\frac{1}{p}}. \tag{37}$$

Iterating this estimate as many times as necessary, and choosing appropriate test functions at each stage (see below), gives us the following theorem

Theorem 1.8 (First ϵ-Regularity Theorem) *Assume $\triangle f + fu \geq 0$, $f \geq 0$, $f \in L^p_{loc}$ for some $p > 1$, $u \in L^{\frac{n}{2}}_{loc}$, and q is any number $q \in (p, \infty)$. Then there exist constants $\epsilon = \epsilon(p, q, C_n(B_r)) > 0$ and $C = C(p, q, C_n(B_r), r) < \infty$ so that*

$$\int_{B_r} u^{\frac{n}{2}} \leq \epsilon \tag{38}$$

implies

$$\left(\int_{B_{r/2}} f^q\right)^{\frac{1}{q}} \leq C\left(\int_{B_r} f^p\right)^{\frac{1}{p}} \tag{39}$$

In other words, $f \in L^p_{loc}$, $p > 1$ and $u \in L^{\frac{n}{2}}_{loc}$ implies $f \in L^q_{loc}$ for all $q \in (p,\infty)$. However both $\epsilon = \epsilon(p,q,C_n)$ and $C = C(p,q,C_n,r)$ deteriorate as $q \to \infty$ (see the exercises), so we do not obtain apriori L^∞ bounds.

1.6.2 Cutoff Functions

It is worthwhile to formalize the choice of test functions necessary to the proof of Theorem 1.8. The test functions we construct below are called *cutoff functions*.

Let $m \in M^n$ be any point, and let $r = dist(m,\cdot)$ be the distance function from m. Distance functions are not smooth, but are Lipschitz and therefore weakly differentiable. Since we encounter at worst gradients of η in our integral estimates, the fact that our "test functions" are just $C^{0,1}$ and not C^∞ is no bother to us.

Given some radius $r_0 \in (0,\infty)$, we may wish to have a test function that is zero outside $B_{r_0}(m)$, is 1 inside $B_{r_0/2}(m)$ and has bounded gradient in the intermediate annulus $B_{r_0}(m) \setminus B_{r_0/2}(m)$. Then define

$$\eta(r) = \begin{cases} 1 & , r \in [0, r_0/2] \\ 2 - \frac{2}{r_0}r & , r \in (r_0/2,\, r_0) \\ 0 & , r \in [r_0,\infty) \end{cases} \tag{40}$$

We have $\nabla\eta = \eta'\nabla r$ and so this function is Lipschitz, and $|\nabla\eta| = |\eta'| \leq \frac{2}{r_0}$ (a.e.), as the gradient of a distance functions has norm 1 (a.e.).

It is also sometimes necessary to have a variety of cutoff functions, with the collection of cutoff functions operating on progressively smaller nested balls, say. To this end consider the functions

$$\eta_i(r) = \begin{cases} 1 & , r \in [0, (2^{-1}+2^{-i-2})r_0] \\ 1 - \frac{2^{i+1}}{r_0}\left(r - (2^{-1}-2^{-i-2})r_0\right) & , r \in \left((2^{-1}+2^{-i-2})r_0,\, (2^{-1}+2^{-i-1})r_0\right) \\ 0 & , r \in [(2^{-1}+2^{-i-1})r_0, \infty). \end{cases} \tag{41}$$

The gradients are

$$|\nabla\eta_i| = \begin{cases} 0 & , r \in [0, (2^{-1}+2^{-i-2})r_0] \\ \frac{2^{i+1}}{r_0} & , r \in \left((2^{-1}+2^{-i-2})r_0,\, (2^{-1}+2^{-i-1})r_0\right) \\ 0 & , r \in [(2^{-1}+2^{-i-1})r_0, \infty). \end{cases} \tag{42}$$

Notice that the supports $supp|\nabla_i|$ are non-overlapping.

1.6.3 Nash-Moser Iteration and the L^∞ estimates

In addition to $f \in L^p$ for some $p > 1$, now assume u is in a strictly larger L^p space than $p = \frac{n}{2}$. We shall assume that $u \in L^{p\gamma}$. (Aside: $u \in L^{p+\delta}$ for any $\delta > 0$ is sufficient to obtain all our conclusions; we refer you to the sources (eg. Gilbarg-Trudinger) for the details.)

Beginning with (34), we observe $\frac{1}{\gamma^2} + \frac{2}{n} + \frac{2}{n}\frac{1}{\gamma} = 1$, which allows us to use Hölder's inequality with greater subtlety:

$$\begin{aligned} & C_n(supp\,\eta)^2 \left(\int \eta^{2\gamma} f^{p\gamma} \right)^{\frac{1}{\gamma}} \\ & \quad \le\; 2\left(1 + \left(\frac{p}{p-1}\right)^2\right) \int |\nabla\eta|^2 f^p \;+\; \frac{p^2}{p-1} \int \left(\eta^{\frac{2}{\gamma}} f^{p\frac{1}{\gamma}}\right) \cdot \left(\eta^{\frac{4}{n}} f^{p\frac{2}{n}}\right) \cdot u \\ & \quad \le\; 2\left(1 + \left(\frac{p}{p-1}\right)^2\right) \int |\nabla\eta|^2 f^p \;+\; \frac{p^2}{p-1} \left(\int \eta^{2\gamma} f^{p\gamma} \right)^{\frac{1}{\gamma^2}} \left(\int \eta^2 f^p \right)^{\frac{2}{n}} \left(\int_{supp\,\eta} u^{\frac{n}{2}\gamma} \right)^{\frac{2}{n}\frac{1}{\gamma}} \end{aligned} \tag{43}$$

An application of Young's inequality followed by a mild rearrangement of terms gives

$$\begin{aligned} & \left[C_n(supp\,\eta)^2 \;-\; \frac{1}{\gamma} \left(\int_{supp\,\eta} u^{\frac{n}{2}\gamma} \right)^{\frac{2}{n}} \right] \left(\int \eta^{2\gamma} f^{p\gamma} \right)^{\frac{1}{\gamma}} \\ & \qquad \le\; 2\left(1 + \left(\frac{p}{p-1}\right)^2\right) \int |\nabla\eta|^2 f^p \;+\; p^{\frac{n}{2}} \left(\frac{p}{p-1}\right)^{\frac{n}{2}} \frac{2}{n} \int \eta^2 f^p. \end{aligned} \tag{44}$$

The benefit is that $\int u^{\frac{n}{2}\gamma}$ must now be small only compared to C_n^2, not p. Thus, even though the right side deteriorates (goes to infinity) like $p^{\frac{n}{2}}$, at least there is *hope* for getting something of value as $p \to \infty$. From now on, assume $p \ge 2$, so $\frac{p}{p-1} \le 2$. We obtain our second, more powerful, bootstrap inequality:

Lemma 1.9 (Second Bootstrapping Inequality) *Suppose f satisfies $\triangle f + uf \ge 0$ where $f \in L^p$ for any $p \ge 2$, and $u \in L^{\frac{n}{2}\gamma}$. If $\int_{supp\,\eta} u^{\frac{n}{2}\gamma} \le \frac{1}{2\gamma} C_n(\mathrm{supp}\ \eta)^2$ then*

$$\left(\int \eta^{2\gamma} f^{p\gamma} \right)^{\frac{1}{p\gamma}} \;\le\; C(n)^{\frac{1}{p}} C_n(supp\,\eta)^{\frac{2}{p}} \left[\sup|\nabla\eta|^2 + p^{\frac{n}{2}} \sup \eta^2 \right]^{\frac{1}{p}} \left(\int_{supp\,\eta} f^p \right)^{\frac{1}{p}}. \tag{45}$$

Notice that $p^{\frac{n}{2p}}$ is uniformly bounded. This gives us the possibility of iteration without the deterioration of the estimate that we saw above.

Let η_i, $i \in \{0, 1, 2, \dots\}$ be the i^{th} cutoff function constructed above, and set $p_i = p\gamma^i$. Also, abbreviate $C_n = C_n(B_{r_0})$. Then Lemma 1.9 gives us

$$\left(\int_{B_{(2^{-1}+2^{-i-2})r_0}} f^{p_{i+1}} \right)^{\frac{1}{p_{i+1}}} \;\le\; C(n)^{\frac{1}{p_i}} C_n^{\frac{2}{p_i}} \left[(2^i r_0)^2 + p_i^{\frac{n}{2}} \right]^{\frac{1}{p_i}} \left(\int_{B_{(2^{-1}+2^{-i-1})r_0}} f^{p_i} \right)^{\frac{1}{p_i}}. \tag{46}$$

This can be iterated all the way down, to obtain

$$\left(\int_{B_{(2^{-1}+2^{-i-2})r_0}} f^{p_{i+1}}\right)^{\frac{1}{p_{i+1}}} \leq \left[\prod_{k=0}^{i} C(n)^{\frac{1}{p_k}} C_n^{\frac{2}{p_k}} \left[(2^k r_0)^2 + p_k^{\frac{n}{2}}\right]^{\frac{1}{p_k}}\right] \left(\int_{B_{r_0}} f^p\right)^{\frac{1}{p}}. \quad (47)$$

Theorem 1.10 (Epsilon-Regularity) *Suppose $f \geq 0$ weakly satisfies $\triangle f + uf \geq 0$, and assume $f \in L^2_{loc}$ and $u \in L^{\frac{n}{2}\gamma}_{loc}$. Given any radius $r_0 > 0$ and point $m \in M^n$, and abbreviating $C_n(B_{r_0}(m))$ by C_n, there exist constants $\epsilon = \epsilon(n, C_n)$ and $C = C(n, C_n, r_0)$ so that if*

$$\int_{B_{r_0}(m)} u^{\frac{n}{2}\gamma} \leq \epsilon \quad (48)$$

then

$$\sup_{B_{r_0/2}(m)} |f| \leq C \left(\int_{B_{r_0}(m)} f^2\right)^{\frac{1}{2}} \quad (49)$$

Proof. One simply checks that

$$\left[\prod_{k=0}^{i} C(n)^{\frac{1}{p_k}} C_n^{\frac{2}{p_k}} \left[(2^k r_0)^2 + p_k^{\frac{n}{2}}\right]^{\frac{1}{p_k}}\right] = \left[\prod_{k=0}^{i} C(n)^{\frac{1}{p\gamma^k}} C_n^{\frac{2}{p\gamma^k}} \left[(2^k r_0)^2 + p^{\frac{n}{2}} \gamma^{k\frac{n}{2}}\right]^{\frac{1}{p\gamma^k}}\right] \quad (50)$$

is uniformly bounded independently of i (see exercises). Setting $p = 2$ and taking $i \to \infty$ on both sides of (47) gives the result. □

This method of finding apriori L^∞ control after assuming just L^2 control is called Moser iteration or Nash-Moser iteration, depending on whether you're an analyst or a geometric analyst.

1.7 Suggested Problems for 1/28

12) Supply a proof for Theorem 1.8. Give explicit estimates for ϵ and C in terms of p, q, $C_n(B_r)$, and r.

13) By considering functions $f : \mathbb{R}^n \to \mathbb{R}$ of the kind $f = r^{-k} (\log r)^{-l}$, show that there exist functions f, u that satisfy the hypotheses of Theorem 1.8 but so that $f \notin L^\infty_{loc}$. Conclude that Theorem 1.8 is optimal. (Hint: computing Laplacians should be easy after you compute $\triangle r$ and verify the identity $\triangle f(r) = f''(r) + f'(r)\triangle r$.)

14) Explicitly estimate $\prod_{k=0}^{\infty} C(n)^{\frac{1}{\gamma^k}} C_n^{\frac{2}{\gamma^k}} \left[(2^k r_0)^2 + \gamma^{k\frac{n}{2}}\right]^{\frac{1}{\gamma^k}}$ in terms of n, C_n and r_0, thereby completing the proof of Theorem 1.10. You could also try leaving $p > 1$ and estimating the product in terms of n, C_n, r_0, and p, thereby bounding $|f|$ in terms of any $\left(\int f^p\right)^{\frac{1}{p}}$, $p > 1$, instead of just $\left(\int f^2\right)^{\frac{1}{2}}$.

1.8 Variational Problems and Euler-Lagrange Equations

1.8.1 Basic Quadratic Functionals and Linear Euler-Lagrange Equations

Let (M^n, g) be a Riemannian manifold, which needn't be compact. If T is any $W^{1,2}$ tensor, consider the quadratic functional

$$\mathcal{F}(T) \;=\; \int_{M^n} |\nabla T|^2 \, dVol. \tag{51}$$

One is sometimes interested in finding minimizers (or more generally extremizers) of such a functional. To do this, we consider variations $T_t = T + tH$ of the tensor T and compute the first order effect on $\mathcal{F}$ (when M^n is non-compact it is normal to put a restriction on H, namely that $H \in W^{1,2}_0$ or $H \in C^\infty_c$ even, in order that we can use Stokes' theorem). If first derivatives of $\mathcal{F}(T_t)$ vanish regardless of what variation H is chosen, then the tensor T is said to be a critical point for $\mathcal{F}$. Then if T is an extremizer, then we compute

$$\begin{aligned} 0 \;&=\; \frac{d}{dt}\Big|_{t=0} \mathcal{F}(T_t) \;=\; \frac{d}{dt}\int |\nabla T_t|^2 \\ &=\; 2\int \langle \nabla T,\, \nabla H\rangle \;=\; -2\int \langle \triangle T,\, H\rangle\,. \end{aligned} \tag{52}$$

Since H is arbitrary, we can choose $H = \eta \triangle T$, where $\eta \geq 0$ is any test function, to obtain the requirement that $\triangle T = 0$. One easily sees that indeed $\triangle T = 0$ if and only if T extremizes $\mathcal{F}$. In general, the pointwise equations one obtains as a requirement for a quantity to extremize a functional are called the *Euler-Lagrange* equations of that functional.

There is a more abstract way to look at this process. Let $\mathcal{T}$ be the vector space of all $W^{1,2}$ sections of tensors on M^n; this is a Banach space. Then $\mathcal{F} : \mathcal{T} \longrightarrow \mathbb{R}$ is a map from one Banach space to another, and more specifically, it is just a function on the (infinite dimensional) space $\mathcal{T}$. As such, we should be able to linearize it (that is, find its "gradient"). This linearization at $T \in \mathcal{T}$ is called its *Frechet derivative* of $\mathcal{F}$ at T, and is sometimes denoted

$$\mathcal{DF}\big|_T \;:\; W^{1,2}_0 \;\longrightarrow\; \mathbb{R}. \tag{53}$$

The first derivative of $\mathcal{F}$ at T in the "direction" of H is

$$\mathcal{DF}\big|_T(H) \;=\; \frac{d}{dt}\Big|_{t=0} \mathcal{F}(T + tH). \tag{54}$$

We are looking for those points $T \in \mathcal{T}$ at which the linearized operator $\mathcal{DF}|_T$ vanishes. Unfortunately we won't have the time to explore this abstract point of view much further than this.

1.8.2 Spaces of Riemannian Metrics, and the Riemannian Moduli Space

We will be applying functionals to the Riemannian structure itself, so we will have to consider the appropriate space of Riemannian structures. So consider the *space of Riemannian metrics*

$$Met(M^n) \;=\; \left\{ g \in \bigotimes{}^2 T^*M \;\middle|\; g \text{ is symmetric and positive definite} \right\} \tag{55}$$

Obviously if $\varphi : M^n \to M^n$ is any diffeomorphism the metric pulls back. Thus the diffeomorphism group $Diff(M^n)$ acts on $Met(M^n)$, giving the so-called *moduli space of Riemannian metrics*

$$Mod(M^n) \;=\; Met(M^n) \,/\, Diff(M^n). \tag{56}$$

The space $Met(M^n)$ is a cone, not a vector space, and is infinte dimensional. The group $Diff(M^n)$ is an extremely complicated infinite dimensional Lie group. It is non-compact, and otherwise very difficult to understand. Thus $Mod(M^n)$ is also difficult to directly understand. Yet is is our fundamental object, as the curvature functionals are invariant under diffeomorphism and therefore pass from maps on $Met(M^n)$ to maps on $Mod(M^n)$. Indeed we may regard Rm as an operator $Rm : \bigodot^2 \to \bigwedge^2 \odot \bigwedge^2$ that is diffeomorphism-invariant:

$$Rm(\varphi^* g) \;=\; \varphi^* \left(Rm(g) \right). \tag{57}$$

1.8.3 Riemannian Functionals

"Optimal" metrics in Riemannian geometry can also be understood as minimizers of certain "energy" functionals. The first is the Hilbert functional (aka the total scalar curvature functional):

$$\mathcal{H}(M^n,\, g) \;=\; \int_{M^n} R \, dVol. \tag{58}$$

This is scale invariant only in dimension $n = 2$, in which case it is related to the Euler constant. To fix this we can normalize by volume

$$\overline{\mathcal{H}}(M^n,\, g) \;=\; \frac{1}{Vol(M^n)^{\frac{n-2}{n}}} \int_{M^n} R \, dVol \tag{59}$$

or we could take R itself to some other power

$$\mathcal{R}(M^n,\, g) \;=\; \int_{M^n} |R|^{\frac{n}{2}} \, dVol \tag{60}$$

Obviously functionals for other curvature quantities also exists:

$$\begin{aligned}
\mathcal{RM}(M^n,\,g) \;&=\; \int_{M^n} |\,\mathrm{Rm}\,|^{\frac{n}{2}}\; dVol\\
\mathcal{RIC}(M^n,\,g) \;&=\; \int_{M^n} |\,\mathrm{Ric}\,|^{\frac{n}{2}}\; dVol\\
\mathcal{W}(M^n,\,g) \;&=\; \int_{M^n} |W|^{\frac{n}{2}}\; dVol.
\end{aligned} \tag{61}$$

In dimension 4 these all take on special interest, as $\frac{n}{2} = 2$. We obtain scale-invariant quadratic curvature functionals

$$\begin{gathered}
\int_{M^n} |\,\mathrm{Rm}\,|^2\; dVol, \quad \int_{M^n} |\,\mathrm{Ric}\,|^2\; dVol, \quad \int_{M^n} |\,\mathring{\mathrm{Ric}}\,|^2\; dVol\\
\int_{M^n} R^2\; dVol, \quad \int_{M^n} |W|^2\; dVol, \quad \int_{M^n} |W^{\pm}|^2\; dVol.
\end{gathered} \tag{62}$$

1.8.4 Riemannian Variational Formulas and non-linear Euler-Lagrange Equations

To compute variations of these functionals, we have to compute variations of the usual Riemannian quantities. Suppose we vary the metric

$$g_t \;=\; g \;+\; th \tag{63}$$

where $h = h_{ij}dx^i \otimes dx^j$ is any symmetric 2-tensor. We have $\frac{dg}{dt} = h$. With the usual formula $\Gamma_{ij}^k = \frac{1}{2}\left(\frac{\partial g_{is}}{\partial x^j} + \frac{\partial g_{js}}{\partial x^i} - \frac{\partial g_{ij}}{\partial x^s}\right) g^{sk}$ we compute

$$\left.\frac{d\Gamma_{ij}^k}{dt}\right|_{t=0} \;=\; \frac{1}{2}\left(\frac{\partial h_{is}}{\partial x^j} + \frac{\partial h_{js}}{\partial x^i} - \frac{\partial h_{ij}}{\partial x^s}\right) g^{sk} \;-\; \frac{1}{2}\left(\frac{\partial g_{is}}{\partial x^j} + \frac{\partial g_{js}}{\partial x^i} - \frac{\partial g_{ij}}{\partial x^s}\right) g^{su} g^{vk} h_{sv}. \tag{64}$$

For theoretical reasons we know that $\dot{\Gamma}_{ij}^k$ is tensorial, so we apply the following standard trick: express $\dot{\Gamma}_{ij}^k$ in a coordinate system in which all Christoffel symbols vanish, then convert all partial derivatives to covariant derivatives. In geodesic normal coordinates, at any single chosen point p we have $g_{ij} = \delta_{ij} + O(2)$ so the $\frac{\partial g}{\partial x}$ terms vanish. Thus we have

$$\begin{aligned}
\left.\frac{d\Gamma_{ij}^k}{dt}\right|_{t=0} \;&=\; \frac{1}{2}\left(\frac{\partial h_{is}}{\partial x^j} + \frac{\partial h_{js}}{\partial x^i} - \frac{\partial h_{ij}}{\partial x^s}\right) g^{sk}\\
\left.\frac{d\Gamma_{ij}^k}{dt}\right|_{t=0} \;&=\; \frac{1}{2}\left(h_{is,j} + h_{js,i} + h_{ij,s}\right) g^{sk}
\end{aligned} \tag{65}$$

Using the standard formulas for $\mathrm{Rm}_{ijk}{}^l$ in terms of the Christoffel symbols, we obtain

$$\begin{aligned}
\frac{d}{dt}\Big|_{t=0}\mathrm{Rm}_{ijk}{}^l(g_t) &= \frac{1}{2}\left(h_{js,ki}+h_{ks,ji}-h_{jk,si}-h_{is,kj}+h_{ks,ij}-h_{ik,sj}\right)g^{sl}\\
\frac{d}{dt}\Big|_{t=0}\mathrm{Ric}_{ij}(g_t) &= \frac{1}{2}\left(h^s{}_{i,js}+h^s{}_{j,is}-h_{jk,s}{}^s-(Tr\,h)_{,ij}\right)\\
\frac{d}{dt}\Big|_{t=0}R(g_t) &= h^{st}{}_{,ts}-\triangle(Tr\,h)\;-\;Ric_{ij}h^{ij}
\end{aligned}\tag{66}$$

From this, along with $\frac{d}{dt}dVol=\frac{1}{2}(Tr\,h)\,dVol$ we can easily compute

$$\frac{d}{dt}\Big|_{t=0}\int R\,dVol \;=\; \int\left\langle Ric\;-\;\frac{1}{2}R\,g\,,\;h\right\rangle dVol \tag{67}$$

so that a stable point for the Hilbert functional occurs when the gravitational tensor vanishes:

$$\begin{aligned}
G_{ij} &\triangleq \mathrm{Ric}_{ij}\;-\;\frac{1}{2}Rg_{ij}\\
G_{ij} &= 0.
\end{aligned}\tag{68}$$

1.9 Suggested Problems for 2/2

15) (Other Elliptic Equations) If (M^n,g) is a compact Riemannian manifold and f,h are integrable functions, determine the Euler-Lagrange equations for the quadratic functional

$$\mathcal{F}(u) \;=\; \int f|\nabla u|^2\;+\;h|u|^2. \tag{69}$$

16) (The p-Laplacian) Determine the Euler-Lagrange equations for the non-quadratic functionals

$$\mathcal{F}(u) \;=\; \int|\nabla u|^p \tag{70}$$

where $p>1$ and, as usual, we take (M^n,g) to be compact. Find the Euler-Lagrange equations for this functionals; these are known as the *p-Laplace equations.* Obviously the 2-Laplacian is just the Laplacian. In the limiting cases $p=1$ and $p=\infty$, what are reasonable notions of the 1- and ∞-Laplacians?

17) Consider the Riemannian metric on $\mathbb{R}^2$

$$g_{ij} \;=\; \frac{4}{(1+r^2)^2}\,\delta_{ij}. \tag{71}$$

This pulls back to $\mathbb{S}^2$, under standard streographic projection, to the metric of constant curvature $+1$. Given any constant $\Lambda>0$, we have diffeomorphisms $\mathbb{R}^2\to\mathbb{R}^2$ given by $(x, y) \mapsto (\Lambda x, \Lambda y)$. Prove that this lifts to a diffeomorphism on $\mathbb{S}^2$ (obviously this map is differentiable away from the north pole; you have to verify it is differentiable at the north pole as well). The metric on $\mathbb{S}^2$ therefore transforms via diffeomorphism by following this proceedure: pushforward under standard stereographic projection, pullback under coordinate multiplication by Λ, and then pullback under standard stereographic projection. Gaussian curvature is a diffeomorphism invariant, so the sectional curvatures of these metrics remains $+1$ (this can also be verified by direct computation). By letting $\Lambda \to 0$ or $\Lambda \to \infty$, show that the set of round metrics of constant curvature $+1$ is not compact the C^∞ (or C^0) topology. Conclude that the diffeomorphism group of $\mathbb{S}^2$ is non-compact.

18) Show that any linear fractional transformation on $\mathbb{R}^2 \cup \{*\}$ lifts to a diffeomorphism on $\mathbb{S}^2$, so that $PSL(2,\mathbb{R}) \subset Diff(\mathbb{S}^2)$. With this, find a sequence of diffeomorphisms $\varphi_i : \mathbb{S}^2 \to \mathbb{S}^2$ such that the sequence of pullbacks $\varphi_i^*(g)$ of a round metric g converges *pointwise* to zero at every point of $\mathbb{S}^2$.

19) Show that the functionals $\int R^k \, dVol$ are diffeomorphism invariant, and therefore indeed pass to functionals $Mod(M^n) \to \mathbb{R}$.

20) Prove formally that each functional is a map $Mod(M^n) \to \mathbb{R}$.

21) Verify the formulas in (66).

22) Show that the Euler-Lagrange equations for the normalized total scalar curvature functional are $\mathrm{Ric}_{ij} - \frac{R}{n} g_{ij} = 0$.

23) Find the Euler-Lagrange equations for the quadratic scalar curvature functional. In dimension other than 4, normalize this functional, and re-compute the Euler-Lagrange equations.

24) (Einstein's Cosmological Constant) If $\Lambda : M^n \to \mathbb{R}$ is a function, we have the Hilbert functional with a cosmological term: $\int (R - \Lambda) dVol$. Find the Euler-Lagrange equations. Show that there exist non-Ricci flat solutions, and that for these solutions Λ must be constant. Such manifolds are called Einstein manifolds.

1.10 Elliptic Equations in the Riemann Tensor and Geometric Apriori Estimates

We have seen that the half-dimension L^p norms play two important roles that appear, at first to be independent. First, the apriori estimates for $\triangle f \geq uf$ require $u \in L^p_{loc}$ for any $p > \frac{n}{2}$, with $p = \frac{n}{2}$ being the critical case where a weak conclusion holds (in the case $p < \frac{n}{2}$ no conclusions at all are possible). Second, the $L^{\frac{n}{2}}$ Riemannian norms are *scale invariant*, and therefore their values carry intrinsic geometric meaning, as opposed to L^p norms for other values of p, which can be made to take any value whatever simply by scaling the metric.

Here we shall see how these two roles combine to produce a remarkable regularity theory for critical metrics (for simplicity we restrict ourselves to Einstein metrics).

1.10.1 Derivative Commutator Formulas

From basic Riemannian geometry we have the definition of the Riemann tensor as a commutator: if X is any vector field then

$$X^l{}_{,ij} - X^l{}_{,ji} = \mathrm{Rm}_{ijs}{}^l X^s. \tag{72}$$

By computation this can be extended to other tensors:

$$T^{l_1\ldots l_s}{}_{,ij} - T^{l_1\ldots l_s}{}_{,ji} = \mathrm{Rm}_{ijs}{}^{k_1}T^{sl_2\ldots l_n} + \mathrm{Rm}_{ijs}{}^{l_2}T^{l_1 s\ldots l_n} + \ldots + \mathrm{Rm}_{ijs}{}^{l_n}T^{l_1\ldots l_{n-1}s} \tag{73}$$

and to covectors $\eta = \eta_i dx^i$

$$\begin{aligned} \eta_{k,ij} - \eta_{k,ji} &= -\mathrm{Rm}_{ijk}{}^s\eta_s \\ T_{k_1\ldots k_n,ij} - \eta_{k,ji} &= -\mathrm{Rm}_{ijk_1}{}^sT_{sk_2\ldots k_n} - \mathrm{Rm}_{ijk_2}{}^sT_{k_1 s\ldots k_n} - \ldots - \mathrm{Rm}_{ijk_n}{}^sT_{k_1\ldots k_{n-1}s}. \end{aligned} \tag{74}$$

This can be extended to tensors of any type. For instance

$$T_a{}^{bc}{}_{,ij} - T_a{}^{bc}{}_{,ji} = \mathrm{Rm}_{ija}{}^sT_s{}^{bc} - \mathrm{Rm}_{ijs}{}^bT_a{}^{sc} - \mathrm{Rm}_{ijs}{}^cT_a{}^{bs} \tag{75}$$

1.10.2 The Second Bianchi Identity and the Riemann Tensor

Both the first and second Bianchi identities can be derived from the diffeomorphism invariance of the Riemann tensor (Reference: J. Kazdan, 1981). If we regard $\mathrm{Rm} = \mathrm{Rm}(g)$ as a second order differential operator on the metric g, then diffeomorphism invariance takes the form

$$\mathrm{Rm}(\varphi^*g) = \varphi^*(\mathrm{Rm}(g)) \tag{76}$$

where $\varphi : M^n \to M^n$ is any diffeomorphism and φ^* is the pullback. Letting φ_t be a time-dependent family of diffeomorphisms with a well-chosen variational field, then taking a Lie derivative on both sides of (76) gives the second Bianchi identity (see exercises). A similar process gives the first Bianchi identity. In this light the Bianchi identities are understood as Leibniz rules.

Alternatively, the second Bianchi identity identity can be computed directly from the definition of Rm, where it ultimately follows from the Jacobi idenity, which itself is both an expression of diffeomorphism invariance (of the bracket), and also a Leibniz rule.

The 2nd Bianchi identity can be expressed in either of the following two ways:

$$\begin{aligned} \mathrm{Rm}_{ijkl,m} + \mathrm{Rm}_{ijlm,k} + \mathrm{Rm}_{ijmk,l} &= 0 \\ \mathrm{Rm}_{ijkl,m} + \mathrm{Rm}_{jmkl,i} + \mathrm{Rm}_{mikl,j} &= 0. \end{aligned} \tag{77}$$

Tracing once gives

$$\mathrm{Ric}_{ij,k} - \mathrm{Ric}_{ik,j} = \mathrm{Rm}_{sijk}{}^{,s} \tag{78}$$

Tracing again gives

$$R_{,k} = 2\mathrm{Ric}_{sk}{}^{,s}. \tag{79}$$

1.10.3 The Laplacian of the Riemann Tensor

Combining the 2nd Bianchi identity with the commutator formula, we can obtain and expression for the Laplacian of the full Riemann tensor. First using the 2nd Bianchi idenitity we get

$$\begin{aligned}(\triangle \mathrm{Rm})_{ijkl} &\triangleq g^{st}\mathrm{Rm}_{ijkl,st} \\ &= -g^{st}\mathrm{Rm}_{ijls,kt} - g^{st}\mathrm{Rm}_{ijsk,lt}\end{aligned} \tag{80}$$

Using the commutator formula on both terms gives

$$\begin{aligned}(\triangle \mathrm{Rm})_{ijkl} &= -g^{st}\mathrm{Rm}_{ijls,tk} - g^{st}\mathrm{Rm}_{ijsk,tl} \\ &\quad + g^{st}\mathrm{Rm}_{kti}{}^{u}\mathrm{Rm}_{ujls} + \ldots\end{aligned} \tag{81}$$

where "…" indicates a further seven quadratic monomials in the Riemann tensor. Whenever the exact expression doesn't matter, we shall abbreviate any linear combination of tensor products and traces of a tensor T with a tensor S simply by $S * T$. To (81), apply the 2nd Bianchi identity again to get

$$\begin{aligned}(\triangle \mathrm{Rm})_{ijkl} &= g^{st}\mathrm{Rm}_{jtls,ik} + g^{st}\mathrm{Rm}_{tils,jk} + g^{st}\mathrm{Rm}_{jtsk,il} + g^{st}\mathrm{Rm}_{tisk,jl} + \mathrm{Rm}*\mathrm{Rm} \\ &= -\mathrm{Ric}_{jl,ik} + \mathrm{Ric}_{il,jk} + \mathrm{Ric}_{jk,il} - \mathrm{Ric}_{ik,jl} + \mathrm{Rm}*\mathrm{Rm}\end{aligned} \tag{82}$$

The second derivative terms are now entirely on various copies of the Ricci tensor. Schematically expressed, we have

$$\triangle \mathrm{Rm} = \nabla^2 \mathrm{Ric} + \mathrm{Rm}*\mathrm{Rm}. \tag{83}$$

In the $n \geq 3$ Einstein case we have $\mathrm{Ric} = \lambda g$ where $\lambda = const$, and so we obtain the tensoral elliptic equation $\triangle \mathrm{Rm} = \mathrm{Rm}*\mathrm{Rm}$. This is difficult to deal with using the above elliptic theory, however, because it was developed for functional Laplace equations of the sort $\triangle f \geq uf$. We therefore resort to the so-called *Kato inequality*: for any tensor T we have

$$|\nabla|T|| \leq |\nabla T|. \tag{84}$$

Then the two identities

$$\begin{aligned}\frac{1}{2}\triangle|T|^2 &= \langle \triangle T, T\rangle + |\nabla T|^2 \\ \frac{1}{2}\triangle|T|^2 &= |T|\triangle|T| + |\nabla|T||^2\end{aligned} \tag{85}$$

along with the Kato inequality give

$$|T|\triangle|T| \geq \langle \triangle T, T\rangle. \tag{86}$$

Using $T = \mathrm{Rm}$ and $|\langle \mathrm{Rm}*\mathrm{Rm}, \mathrm{Rm}\rangle| \geq -C|\mathrm{Rm}|^3$ (from Cauchy-Schwartz) we get the equation

$$\triangle|\mathrm{Rm}| \geq C(n)|\mathrm{Rm}|^2 \tag{87}$$

which holds strongly away from $|\mathrm{Rm}| = 0$ and holds weakly everywhere.

1.10.4 Consequences of the Einstein Condition

We have seen that the Einstein condition gives the weak elliptic inequality

$$\triangle u \;\geq\; -u^2 \tag{88}$$

where $u = C|\operatorname{Rm}|$. Using the L^p theory above, where $f = u$, gives the statement that there are constants $\epsilon = \epsilon(n, C_s)$, $= C(n, C_s, r)$ so that if

$$\left(\int_{B_r(m)} |\operatorname{Rm}|^{\frac{n}{2}}\right)^{\frac{2}{n}} \;\leq\; \epsilon \tag{89}$$

then

$$\left(\int_{B_{\frac{3}{4}r}(m)} |\operatorname{Rm}|^{\gamma\frac{n}{2}}\right)^{\frac{2}{n}\frac{1}{\gamma}} \;\leq\; C\left(\int_{B_r(m)} |\operatorname{Rm}|^{\frac{n}{2}}\right)^{\frac{2}{n}} \tag{90}$$

(this is from Theorem 1.8). It can be shown that the r–dependency of C is $r^{-\frac{4}{n}}$ (see exercises). But then with a uniform local estimate of the $L^{\frac{n}{2}\gamma}$-norm of u we can actually use the more powerful ϵ-regularity theorem (Theorem 1.10) to obtain

$$\left(\int_{B_r(m)} |\operatorname{Rm}|^{\frac{n}{2}}\right)^{\frac{2}{n}} \;\leq\; \epsilon \quad\Longrightarrow\quad \sup_{B_{\frac{1}{2}r}(m)} |\operatorname{Rm}| \;\leq\; C\left(\int_{B_r(m)} |\operatorname{Rm}|^{\frac{n}{2}}\right)^{\frac{2}{n}} \tag{91}$$

In the exercises, you'll show that the r-dependence in C is actually r^{-2}.

This remarkable theorem, due independently to Anderson and Bando-Kasue-Nakajima, gives local apriori control over the <u>*pointwise*</u> norm of the Riemann tensor in terms of local $L^{\frac{n}{2}}$ norms of $|\operatorname{Rm}|$.

1.11 Problems for 2/4

25) In this exercise we'll work through how the diffeomorphism invariance of the various curvature tensors imply the doubly-traced Second Bianci identity (79). Let φ_t be a family of diffeomorphisms with variation field X. Consider the expression $R(\varphi_t^* g) = \varphi_t^*(R(g))$, and take the derivative $\frac{d}{dt}\big|_{t=0}$ on both sides. On the right side, obtain simply $X(R)$, and on the left side use the variation formulas for scalar curvature from (66) along with the identity $(\mathcal{L}_X g)_{ij} = (X_{i,j} - X_{j,i})$. Finally, apply a commutator formula to obtain (66).

26) (Harder, but not terrible) Use the process from Problem 25 and the variation formula for the Riemann tensor to derive the traced 2nd Bianci and finally the full 2nd Bianchi identity.

27) Derive the exact expression for $\mathrm{Rm} * \mathrm{Rm}$ in (83). Then derive an explicit estimate for $C = C(n)$ in the weak inequality $\triangle|\,\mathrm{Rm}\,| \geq -C|\,\mathrm{Rm}\,|^2$.

28) Prove the Kato inequality.

29) Run through the "First Bootstrapping Inequality" (Lemma 1.7) with a proper choice of η to obtain

$$\left(\int_{B_{\frac{3}{4}r}(m)} |\,\mathrm{Rm}\,|^{\gamma\frac{n}{2}}\right)^{\frac{2}{n}\frac{1}{\gamma}} \;\leq\; Cr^{-\frac{4}{n}}\left(\int_{B_r(m)} |\,\mathrm{Rm}\,|^{\frac{n}{2}}\right)^{\frac{2}{n}} \tag{92}$$

for $C = C(n, C_n)$, under the condition that $\left(\int_{B_r(m)} |\,\mathrm{Rm}\,|^{\frac{n}{2}}\right)^{\frac{2}{n}} \leq \epsilon$.

30) Use your result from Problem 29, along with a slightly more careful Moser iteration process from Theorem 1.10 to obtain

$$\sup_{B_{\frac{1}{2}r}(m)} |\,\mathrm{Rm}\,| \;\leq\; Cr^{-2}\left(\int_{B_r(m)} |\,\mathrm{Rm}\,|^{\frac{n}{2}}\right)^{\frac{2}{n}} \tag{93}$$

for $C = C(n, C_n)$, under the condition that $\left(\int_{B_r(m)} |\,\mathrm{Rm}\,|^{\frac{n}{2}}\right)^{\frac{2}{n}} \leq \epsilon$.

1.12 Harmonic Coordinates

The material in this section is due to DeTurck-Kazdan.

1.12.1 Definition and First Uses

One interested in what local coordinates are "best." Obviously what "best" means will vary from situation to situation , and geodesic normal coordinates are particularly useful in certain cases, as we already saw when we computed variation formulas.

However, for investigating regularity issues, geodesic normal coordinates are not good. If a metric is only $C^{k,\alpha}$, for instance, then the exponential map causes a derivative loss when constructing the coordinate frame, so, in fact, a $C^{k,\alpha}$ metric will not appear even to be $C^{k,\alpha}$ in its own exponential normal coordinate chart (see the DeTurck-Kazdan paper for details). So if the metric is $C^{k,\alpha}$, what sort of coordinate chart will show this?

An *harmonic coordinate chart* is simply a coordinate chart $\{x^1, \ldots, x^n\}$ in which each coordinate is harmonic:

$$\triangle\, x^i \;=\; 0. \tag{94}$$

Using $\triangle f = g^{ij}\frac{\partial^2 f}{\partial x^i \partial x^j} - \Gamma^k \frac{\partial f}{\partial x^k}$, we easily see that $\{x^1, \dots, x^n\}$ is an harmonic chart if and only if each of the traced Christoffel symbols $\Gamma^k = g^{ij}\Gamma_{ij}^k$ vanishes. A theorem of DeTurck-Kazdan states that, for regularity issues, this coordinate frame is the best:

Theorem 1.11 (DeTurck-Kazdan) *Let (M^n, g) be a differentiable manifold along with a metric of class $C^{k,\alpha}$. Assume T is a tensor of class $C^{l,\beta}$. If $\{x^1, \dots, x^n\}$ is an harmonic coordinate chart, then the component functions of T will be differentiable of class $C^{m,\gamma}$ where $m + \gamma = \min\{k + \alpha, l + \gamma\}$.*

In particular, a $C^{k,\alpha}$ metric will appear to be $C^{k,\alpha}$ in its own harmonic coordinate chart.

1.12.2 The Ricci Operator

The Ricci curvature tensor can be considered a non-linear differential operator that takes the metric g as its input and produces a symmetric 2-tensor as its output. In harmonic coordinates this transformation is explicitly a semilinear elliptic differential operator, as we shall see.

We have

$$\mathrm{Ric}_{ij} = \frac{\partial \Gamma_{ij}^s}{\partial x^s} - \frac{\partial \Gamma_{sj}^s}{\partial x^i} + \Gamma_{tj}^s \Gamma_{is}^t - \Gamma_{ij}^s \Gamma_{ts}^t. \tag{95}$$

We shall be interested exclusively in the second order part, so we shall denote all lower order terms with the expression $Q = Q(\partial g, g)$, which will be quadratic separately in ∂g and in g, and a polynomial of 4th degree in both together. Expanding out the Christoffel symbols and making a cancellation we obtain

$$\mathrm{Ric}_{ij} = -\frac{1}{2} g^{st} \frac{\partial^2 g_{ij}}{\partial x^s \partial x^t} + \frac{1}{2} g^{ts} \left(\frac{\partial^2 g_{it}}{\partial x^s \partial x^j} + \frac{\partial^2 g_{js}}{\partial x^i \partial x^t} - \frac{\partial^2 g_{st}}{\partial x^i \partial x^j} \right) + Q. \tag{96}$$

Next consider the expression $g^{ui}\frac{\partial \Gamma^j}{\partial x^u}$. We have

$$\begin{aligned} g_{ri}\frac{\partial \Gamma^r}{\partial x^j} &= \frac{1}{2} g_{ri} g^{uv} g^{sr} \left(\frac{\partial^2 g_{us}}{\partial x^j \partial x^v} + \frac{\partial^2 g_{vs}}{\partial x^j \partial x^u} - \frac{\partial^2 g_{uv}}{\partial x^j \partial x^s} \right) + Q \\ &= g^{uv} \frac{\partial^2 g_{ui}}{\partial x^v \partial x^j} - \frac{1}{2} g^{uv} \left(\frac{\partial^2 g_{uv}}{\partial x^i \partial x^j} \right) + Q. \end{aligned} \tag{97}$$

Comparing this with (98) we see that, in general,

$$\mathrm{Ric}_{ij} = -\frac{1}{2} g^{st} \frac{\partial^2 g_{ij}}{\partial x^s \partial x^t} + \frac{1}{2} \left(g_{ri} \frac{\partial \Gamma^r}{\partial x^j} + g_{rj} \frac{\partial \Gamma^r}{\partial x^i} \right) + Q. \tag{98}$$

Therefore in an harmonic coordinate system this simplifies to

$$\mathrm{Ric}_{\,ij} = -\frac{1}{2} \triangle_c g_{ij} + Q \tag{99}$$

where $\triangle_c$ indicates the *coordinate Laplacian* $\triangle_c \triangleq g^{st}\frac{\partial^2}{\partial x^s \partial x^t}$.

1.12.3 Bootstrapping Regularity for Einstein Manifolds

Recall that in the case of Einstein metrics in dimension $n \geq 3$, we derived apriori local estimates on the Riemann curvature tensor, obtaining local L^∞ bounds. Via the Jacobi equation, this puts apriori C^2 control on the metric tensor. Now consider, in an harmonic coordinate system, the equation

$$\lambda g_{ij} \;=\; -\frac{1}{2}\triangle_c g_{ij} \;+\; Q, \tag{100}$$

where $\lambda = const$, which comes from (99) and the Einstein condition. Because $g \in C^2$ we have $Q \in C^1$, and that the coefficients on $\triangle_c$ are C^2. The Schauder apriori estimates therefore imply that the components g_{ij} must be $C^{2,\alpha}$ functions. But then $Q \in C^{1,\alpha}$, so the Schauder theory gives $g_{ij} \in C^{3,\alpha}$. Now $Q \in C^{2,\alpha}$, so Schauder implies $g_{ij} \in C^{4,\alpha}$. Continuing in this way, we obtain apriori C^∞ estimates for g_{ij}.

Indeed, as g satisfies an elliptic equation whose coefficients are themselves determined by g, it is possible to prove that the g_{ij} functions must be C^ω (real-analytic). Thus an Einstein metric is necessarily real-analytic.

This concludes our investigation of Einstein manifolds from an elliptic-analytic point of view.

1.13 Problems for 2/9

31) In an arbitrary coordinate system, show that the traced Christoffel symbols Γ^k are logarithmic derivatives of the volume forms $\sqrt{\det g}$.

32) (Modified harmonic coordinates) Suppose $\triangle x^i + b^{ij}_k \frac{\partial x^j}{\partial x^k} + c^i_j x^j + f^i = 0$. Write down some conditions on the functions b^{ij}_k, c^i_j, f^i so that the Ricci operator is still elliptic semilinear in the resulting gauge.

1.14 Introduction to Basic Complex Geometry

1.14.1 Almost Complex Structures and Initial Constructions

Many approaches to complex geometry exist; in these notes we'll choose an approach that should give differential geometers the gentlest possible introduction. We will regard complex geometry and/or topology to simply be differential geometry and/or topology with an additional differentiable structure, the so-called *almost complex structure* J, which is simply any tensor

$$J : TM \longrightarrow TM \quad \text{such that} \quad J \circ J \;=\; -Id. \tag{101}$$

Note that J can be dualized: if η is any covector then we define $J(\eta)$ to be the covector $\eta \circ J$ (beware: sometimes the convention $J(\eta) = -\eta \circ J$ is used instead).

Because J squares to -1 its eigenvalues are $\pm\sqrt{-1}$, and its eigenvectors come in complex conjugate pairs. If we define the complexified tangent space to be

$$T_{\mathbb{C}}M \triangleq C \otimes TM \tag{102}$$

then we have an eigenspace decomposition of $T_{\mathbb{C}}M$ into two complex vector bundles

$$T_{\mathbb{C}}M = T'_{\mathbb{C}}M \oplus T''_{\mathbb{C}}M \tag{103}$$

called the *holomorphic* and *antiholomorphic* tangent bundles (respectively the $+\sqrt{-1}$ and $-\sqrt{-1}$ eigenspaces of J). We often denote these $T'_{\mathbb{C}}M = T_{(1,0)}M$ and $T''_{\mathbb{C}}M = T_{(0,1)}M$. Similarly the complexified cotangent bundle $T^*_{\mathbb{C}}M = T^{(1,0)}M \oplus T^{(0,1)}M$ splits into eigenspaces. We have projectors

$$\begin{aligned}
\pi_{(1,0)} : T_{\mathbb{C}}M &\longrightarrow T_{(1,0)}M, & \pi_{(1,0)} &= \frac{1}{2}\left(Id - \sqrt{-1}J\right) \\
\pi_{(0,1)} : T_{\mathbb{C}}M &\longrightarrow T_{(0,1)}M, & \pi_{(0,1)} &= \frac{1}{2}\left(Id + \sqrt{-1}J\right) \\
\pi^{(1,0)} : T_{\mathbb{C}}M &\longrightarrow T^{(1,0)}M, & \pi^{(1,0)} &= \frac{1}{2}\left(Id - \sqrt{-1}J\right) \\
\pi^{(0,1)} : T_{\mathbb{C}}M &\longrightarrow T^{(0,1)}M, & \pi^{(0,1)} &= \frac{1}{2}\left(Id + \sqrt{-1}J\right).
\end{aligned} \tag{104}$$

The projectors are complimentary in the sense that $\pi_{(1,0)} \circ \pi_{(0,1)} = \pi_{(0,1)} \circ \pi_{(1,0)} = 0$ and $Id = \pi_{(1,0)} + \pi_{(0,1)}$, and similarly for the cotangent bundle projectors.

A vector field X is a section of the holomorphic vector bundle if $X = \pi_{(1,0)}(X)$, or equivalently if $\pi_{(0,1)}(X) = 0$. Such a vector field X must have the form $X = Y - \sqrt{-1}JY$ for some real vector field Y. Likewise X is a section of the antiholomorphic vector bundle if $X = Y + \sqrt{-1}JY$ for some real field Y.

1.14.2 Integrability and the Nijenhuis Tensor

Because $T_{(0,1)}M$ is a complex vector bundle, it does not make sense to talk about its "integrability" in the sense that in spans any kind of submanifold. Yet it makes complete sense that we can talk about the Frobenius criterion, so we say that $T_{(1,0)}M$ is integrable iff $[T_{(1,0)}M, T_{(1,0)}M] \subseteq T_{(1,0)}M$. Letting $Y - \sqrt{-1}JY$ and $Z - \sqrt{-1}JZ$ be sections of the holomorphic vector bundle, linearity of the bracket gives that

$$[Y - \sqrt{-1}JY,\ Z - \sqrt{-1}JZ] \in T_{(1,0)}M \tag{105}$$

if and only if

$$[Y, Z] - [JY, JZ] + J\left([Y, JZ] - [JY, Z]\right) = 0. \tag{106}$$

It is convenient to define the *Nijenhuis tensor* N_J to be

$$N_J(Y,\,Z)\;=\;[Y,\,Z]\;-\;[JY,\,JZ]\;+\;J\left([Y,\,JZ]\;-\;[JY,Z]\right). \tag{107}$$

Although it may not be obvious, the Nijenhuis tensor is actually linear with respect to multiplying Y or Z by a function (see exercises). Thus the condition for integrability is that $N_J\equiv 0$. An almost complex structure that is integrable is called a *complex structure.*

1.14.3 Example: Orientable Riemannin Manifolds of Dimension 2

If (Σ^2, g) is an oriented manifold with a Riemannian metric, then a Hodge-$*$ exists. On any n-manifold we have $** : \bigwedge^k \to \bigwedge^k$, $** = (-1)^{k(n-k)}$. Therefore on Σ^2 we have $** : T^*\Sigma \to T^*\Sigma$, $** = -1$, is an almost complex structure. Further, since the manifold is Riemannian, the brackets that constitute the Nijenhuis tensor can be evaluated using covariant derivatives. This, combined with the covariant-constancy of the Hodge-$*$, gives in fact $N_*\equiv 0$. Thus $J=*$ is a complex structure on any oriented Riemannian 2-manifold. One can verify that $* : T^*M \to T^*M$ is a conformal invariant, so therefore the natural complex structure obtained from g is also conformally invariant.

Notice that in dimension 2 (and only dimension 2) the Laplace equation $\triangle f = 0$ is also conformally invariant, meaning that if a function f is harmonic with respect to some metric, it remains harmonic in any conformally related metric. This can be verifies as such: $\triangle f\;=$-$*d(*df)$ so therefore $\triangle f = 0$ is the same as $d(*df) = 0$. But the 1-form $*df$ is conformally invariant (because d is independent of the metric and $*$ is conformally invariant), and so the equation $d(*df) = 0$ is conformally invariant.

Given any harmonic function f defined on a simply connected domain $\Omega\subseteq\Sigma^2$, there is a conjugate harmonic function g, defined implicitly by $dg = *df$ (since $*df$ is closed, this follows from the Poincare lemma).

Theorem 1.12 *If f is any harmonic function and g is its conjugate in the sense that $dg = *df$, then $z = g+\sqrt{-1}f$ is a holomorphic function.*

Proof. Compute $\pi^{(0,1)}(dz)$ and verify that it is zero. □

1.14.4 Holomorphic functions, holomorphic coordinate systems, and integrability

A function is called holomorphic if its differential df is a section of the holomorphic cotangent bundle; that is, if $df = \pi^{(1,0)}(df)$.

An excellent question is whether holomorphic functions exist. More broadly, if (M^{2n}, J) is an almost complex manifold, one wishes to know when there exist complex functions

$$z^1 = x^1+\sqrt{-1}y^1,\;\ldots,\;z^n = x^n\;+\;\sqrt{-1}y^n \tag{108}$$

so that the $2n$ many functions $x^1, y^1, \ldots, x^n, y^n$ form a coordinate system, and so that the function $\{z^i\}$ are all holomorphic. When this happens, we say that $\{z^1, \ldots, z^n\}$ form a holomorphic coordinate system.

This question was answered by Newlander-Nirenberg:

Theorem 1.13 (Newlander-Nirenberg) *Let (M^{2n}, J) be an almost complex manifold. Then exists a holomorphic coordinate system $\{z^1, \ldots, z^n\}$ near any point $p \in M$ if and only if $N_J \equiv 0$.*

We shall not attempt any sort of proof of this difficult theorem.

One final word. If $\{z^1, \ldots, z^n\}$ and $\{w^1, \ldots, w^n\}$ are two different holomorphic coordinates systems, then the transition functions $\frac{\partial z^i}{\partial w^j}$ are in fact automatically holomorphic. If $N_J \equiv 0$, the Newlander-Nirenber theorem states that we can cover M^n with domains that each have holomorphic coordinate systems. Since the transitions on the overlaps are automatically holomorphic, this means we have an equivalence amongst three different notions of a *complex manifold*. Namely (M^{2n}, J) is a complex manifold iff the holomorphic tangent bundle $T_{(1,0)}M$ is integrable iff the Nijenhuis tensor N_J vanishes iff M^{2n} can be covered by J-holomorphic coordinate charts with holomorphic coordinate transitions.

1.15 Problems for 2/11

33) Formally show that if a manifold M^n admits an almost complex structure, then n is even.

34) Verify that if X is a section of the holomorphic vector bundle iff $X = Y - \sqrt{-1}JY$ for some real vector field Y.

35) Prove that the Nijenhuis tensor is indeed a tensor. In particular, if f is any function, show $N_J(fY, Z) = fN_J(Y, Z) = N_J(Y, fZ)$.

36) Show that $T_{(1,0)}M$ is integrable if and only if $T_{(0,1)}M$ is integrable.

37) Verify that on oriented Riemann 2-manifolds (Σ^2, g) we have that $*$ is covariant-constant, and that the Nijenhuis tensor N_* actually vanishes.

38) Prove that $* : \bigwedge^1 \to \bigwedge^1$ is a conformal invariant on any oriented Riemannian 2-manifold.

39) If f is any function, show that $df = \pi^{(1,0)} df$ are precisely the Cauchy-Riemann equations. (Hint: do this first for the 2-dimensional case.)

1.16 The Bigraded Exterior Algebra, and the ∂ and $\bar\partial$ Operators

1.16.1 The bigraded exterior algebra

The decomposition of $T^*_{\mathbb{C}}M$ into eigenspaces of the J operator implies a refinement of the usual grading of the exterior algebra.

If J is any almost complex structure, we have separate (complex) vector bundles $T^{(1,0)}M$ and $T^{(0,1)}M$, and so we have separate exterior algebras built over them:

$$\begin{aligned} \bigwedge^{*,0} &= \bigoplus_{k=0}^{n}\bigwedge^{k}T^{(1,0)}M = \bigoplus_{k=1}^{\infty}\bigwedge^{k}T^{(1,0)}M = \bigoplus_{k=1}^{n}\bigwedge^{k,0} \\ \bigwedge^{0,*} &= \bigoplus_{k=0}^{n}\bigwedge^{k}T^{(0,1)}M = \bigoplus_{k=1}^{\infty}\bigwedge^{k}T^{(0,1)}M = \bigoplus_{k=1}^{n}\bigwedge^{0,k} \end{aligned} \tag{109}$$

Of course the full exterior algebra $\bigwedge^* TM$, usually denotes $\bigwedge^{*,*}$ has the following decomposition

$$\begin{aligned} \bigwedge^{*,*} &\triangleq \bigwedge^{*}\left(T^{(1,0)}M \oplus T^{(0,1)}M\right) \\ &= \bigoplus_{k}\left(\bigoplus_{i=0}^{k}\left(\bigwedge^{i}T^{(1,0)}M \otimes \bigwedge^{k-i}T^{(0,1)}M\right)\right). \end{aligned} \tag{110}$$

We denote $\bigwedge^{i,j} = \bigwedge^{i} T^{(1,0)}M \otimes \bigwedge^{j} T^{(0,1)}M$, so that $\bigwedge^{k} = \bigoplus_{i=1}^{k}\bigwedge^{i,k-i}$ and $\bigwedge^{*} = \bigwedge^{*,*} = \bigoplus_{i,j=0}^{n}\bigwedge^{i,j}$.

Define operators

$$\begin{aligned} \partial &: \bigwedge^{k,l} \longrightarrow \bigwedge^{k+1,l}, \qquad \partial\eta = \pi^{(k+1,l)}(d\eta) \\ \bar\partial &: \bigwedge^{k,l} \longrightarrow \bigwedge^{k,l+1}, \qquad \partial\eta = \pi^{(k,l+1)}(d\eta). \end{aligned} \tag{111}$$

The operator $d : \bigwedge^0 \to \bigwedge^1$ obviously decomposes $d = \partial + \bar\partial$. However, this does not hold in general. All we can say is that

$$d : \bigwedge^{k,l} \to \bigwedge^{k+l+1}, \tag{112}$$

but there is no information that $d = \partial + \bar\partial$. For general J, there is no straightforward decomposition for d. **However if J is integrable, the interaction is straightforward**, as we see presently.

Consider a holomorphic coordinate system $\{z^1, \dots, z^n\}$. By Newlander-Nirenberg, this exists *if and only if* J is integrable. The Cauchy-Riemann equations are

$$\bar\partial f = 0, \qquad \text{or} \qquad Jdf = \sqrt{-1}df \tag{113}$$

and therefore $dz^i = \partial z^i$. Newlander-Nirenberg says that $T_{(0,1)}M$ is integrable, and a Frobenius theorem says therefore $d : \bigwedge^{1,0} \to \bigwedge^{2,0} \oplus \bigwedge^{1,1}$ (see exercises) and, but taking complex conjugates, also $d : \bigwedge^{0,1} \to \bigwedge^{0,2} \oplus \bigwedge^{1,1}$. This means, in particular, that for any function f we have

$$0 \;=\; d^2 f \;=\; d(\partial f) \,+\, d(\bar\partial f) \tag{114}$$

But the integrability of J gives that

$$\begin{aligned} d\partial f &\in \bigwedge^{2,0} \oplus \bigwedge^{1,1}, \quad \text{so that} \quad d\partial f = \partial\partial f \,+\, \bar\partial\partial f \\ d\bar\partial f &\in \bigwedge^{0,2} \oplus \bigwedge^{1,1}, \quad \text{so that} \quad d\bar\partial f = \bar\partial\bar\partial f \,+\, \partial\bar\partial f \end{aligned} \tag{115}$$

Matching by type, we conclude that

$$\begin{aligned} \partial\partial : \bigwedge^{0,0} &\longrightarrow \bigwedge^{20}, \qquad \partial\partial \;\equiv\; 0 \\ \bar\partial\bar\partial : \bigwedge^{0,0} &\longrightarrow \bigwedge^{0,2}, \qquad \bar\partial\bar\partial \;\equiv\; 0 \\ \bar\partial\partial \,+\, \partial\bar\partial \;:\; \bigwedge^{0,0} &\longrightarrow \bigwedge^{1,1}, \qquad \bar\partial\partial \,+\, \partial\bar\partial \;\equiv\; 0. \end{aligned} \tag{116}$$

Theorem 1.14 *If J is an integrable complex structure, there is a unique operator $\partial : \bigwedge^{k,l} \to \bigwedge^{k+1,l}$ that satisfies*

1) *(Constant linearity) $\partial(n\eta + m\gamma) \;=\; n\partial\eta \,+\, m\partial\gamma$*
2) *(Leibniz Rule) $\partial(\eta \wedge \gamma) \;=\; \partial\eta \wedge \gamma + (-1)^{deg\,\eta}\eta \wedge \partial\gamma$*
3) *(Flatness) The operator $\partial\partial : \bigwedge^{0,0} \to \bigwedge^{2,0}$ is precisely zero*
4) *(Action on functions) $\partial f = \pi^{1,0} df$.*

Proof. See exercises. ☐

Theorem 1.15 *If J is an integrable complex structure, there is a unique operator $\bar\partial : \bigwedge^{k,l} \to \bigwedge^{k,l+1}$ that satisfies*

1) *(Constant linearity) $\bar\partial(n\eta + m\gamma) \;=\; n\bar\partial\eta \,+\, m\bar\partial\gamma$*
2) *(Leibniz Rule) $\bar\partial(\eta \wedge \gamma) \;=\; \bar\partial\eta \wedge \gamma + (-1)^{deg\,\eta}\eta \wedge \bar\partial\gamma$*
3) *(Flatness) The operator $\bar\partial\bar\partial : \bigwedge^{0,0} \to \bigwedge^{2,0}$ is precisely zero*
4) *(Action on functions) $\bar\partial f = \pi^{0,1} df$.*

Proof. See exercises. □

Theorem 1.16 *If J is an integrable complex structure, then in fact $d : \bigwedge^{k,l} \to \bigwedge^{k+1,l} \oplus \bigwedge^{k,l+1}$ and $d = \partial + \bar{\partial}$.*

Proof. See exercises. □

1.17 Holomorphic Vector Bundles and the $\bar{\partial}$ Operator

1.17.1 Definition of Holomorphic Vector Bundles

A rank k real vector bundle E over M^n can be described via charts $\{U_\alpha\}_\alpha$ that cover M^{2n} (countably many with finite intersections) and "trivializations" $U_\alpha \times \mathbb{R}^k$, along with a collection of "transition functions"

$$\rho_{\beta\alpha} : U_\alpha \cap U_\beta \longrightarrow GL(\mathbb{R}, k) \tag{117}$$

that satisfy the *cocycle conditions* that $\rho_{\alpha\beta} = \rho_{\beta\alpha}^{-1}$ and that on triple intersection $\rho_{\gamma\alpha} = \rho_{\gamma\beta}\rho_{\beta\alpha}$. Then we can form equivalence classes: given $(x, v) \in U_\alpha \times \mathbb{C}^k$ and $(y, w) \in U_\beta \times \mathbb{C}^k$ we say

$$(x, v) \sim (y, w) \quad \text{iff} \quad x = y \text{ and } w = \rho_{\beta\alpha} v. \tag{118}$$

A rank k *complex vector bundle* is the same thing, except now the transitions are

$$\rho_{\beta\alpha} : U_\alpha \cap U_\beta \longrightarrow GL(\mathbb{R}, k) \tag{119}$$

A complex vector bundle is a *holomorphic vector bundle* provided the base space M^n is itself a complex manifold and provided, after choosing a basis for $\mathbb{C}^k$ and therefore being able to describe the $\rho_{\beta,\alpha}$ as matrix-values maps $\rho_{\beta\alpha} : U_\alpha \cap U_\beta \to M^{\mathbb{C}}_{n\times n}$, we have that the transitions are themselves holomorphic:

$$\bar{\partial}\, \rho_{\beta\alpha} = 0. \tag{120}$$

1.17.2 The $\bar{\partial}$ operator on holomorphic vector bundles

In the real case, the exterior derivative exists exclusively on the base manifold. By contrast, in the case of a holomorphic vector bundle E, there does exist a uniquely-defined exterior derivative

$$\bar{\partial} : E \longmapsto T^{(0,1)} M \otimes E \tag{121}$$

(however it will be impossible to iterate this operator, so for instance $\bar{\partial}\bar{\partial}$ will not be defined). If $\{e_1, \ldots, e_k\}$ is a basis for $\mathbb{C}^k$, then any section e can be expressed over trivializations $U_\alpha \times \mathbb{C}^k$ by $e = a^i e_i$ and over $U_\beta \times \mathbb{C}^k$ by $e = b^i f_i$ where the transition is $f_i = (\rho_{\beta\alpha})_i^j e^j$.

However the transition component functions $(\rho_{\beta\alpha})_j^i$ are holomorphic, so by defining

$$\bar{\partial} e \;=\; \bar{\partial}(a^i) \otimes e_i \tag{122}$$

over a trivialization, we see that this definition is invariant over which trivialization is chosen. To wit:

$$\begin{aligned} \bar{\partial} e \;&=\; \bar{\partial}(b^i) \otimes f_i \\ &=\; \bar{\partial}(a^j (\rho_{\beta\alpha})_j^i) \otimes f_i \\ &=\; \bar{\partial}(a^j) \otimes (\rho_{\beta\alpha})_j^i f_i \\ &=\; \bar{\partial}(a^j) \otimes e_i. \end{aligned} \tag{123}$$

Thus we see that the definition of $\bar{\partial}$ is trivialization-independent, so $\bar{\partial}$ exists as an operator on E. A section e of E that has $\bar{\partial} e = 0$ is called a holomorphic section.

The holomorphic tangent bundle $T_{(1,0)}M$ of course has a $\bar{\partial}$ operator. A vector field $\mathcal{X}$ on M that is a section of $T_{(1,0)}M$ is called an *holomorphic vector field* provided $\bar{\partial}\mathcal{X} = 0$.

Obviously there are no holomophic functions on a compact complex manifold except the constants (due to the maximum principle). But there are sometimes holomorphic vector fields, as you'll see in the exercises.

1.18 Problems for 2/16

40) Verify, using a Frobenius theorem and some trivial identities, that $[T_{(1,0)}M, T_{(1,0)}M] \subset T_{(1,0)}M$ if and only if $d : \bigwedge^{1,0} \to \bigwedge^{2,0} \oplus \bigwedge^{1,1}$.

41) Prove Theorems 1.14, 1.15, and 1.16. Hint for Theorem 1.16: compute the sum $\frac{\partial}{\partial z^i} dz^i + \frac{\partial}{\partial \bar{z}^i} d\bar{z}^i$ in terms of the real coframe basis $\{dx^1, dy^1, \ldots, dx^n, dy^n\}$.

42) Suppose (M^n, J) is a complex manifold, and E is a complex k-vector bundle over M^n. Assume that E has an exterior derivative $\bar{\partial} : E \to T^{(0,1)}M \otimes E$ with $\bar{\partial}(f4) = \bar{\partial} f \otimes e + f\bar{\partial} e$ for any section e of E. Prove that, in fact, E is an holomorphic vector bundle.

43) Consider the 2-sphere $\mathbb{S}^2$, with a complex structure J give by the right-hand rule, oriented via the inward pointing normal. Prove that the stereographic projections preserve J. Prove that the vector field $z\frac{d}{dz}$ on $\mathbb{C}$ lifts to a smooth field on $\mathbb{S}^2$; this is called the "Euler field." Prove that it is a non-constant holomorphic vector field on $(\mathbb{S}^, J)$.

44) Suppose X is a smooth field on any complex manifold (M^n, J) that preserves J in the sense that $\mathcal{L}_X J = 0$. Prove that $X - \sqrt{-1}JX$ is a holomorphic vector field. Conversely, if $\mathcal{X}$ is any holomorphic vector field, prove that $\mathcal{L}_{Re(\mathcal{X})} J = \mathcal{L}_{Im(\mathcal{X})} J = 0$.

1.19 Covariant Exterior Differentiation and Curvature of Vector Bundles

For a moment we retreat from complex geometry as such and develop some ideas in differential geometry that will be of use to us, even in the complex case. A connection on a vector bundle E (whether real or complex) is a map

$$\nabla \,:\, E \;\longrightarrow\; T^*M \otimes E \tag{124}$$

that obeys a Leibniz rule: if σ is any section of E and f is any function, then

$$\nabla(f\sigma) = df \otimes E + f\,\nabla\sigma. \tag{125}$$

We can combine any connection ∇ on E with the differential operator d on the base manifold to create *covariant exterior differentiation*, which is often denoted D. We do this by simply requiring that a Leibnz rule hold:

$$\begin{aligned} D \,:\, \bigwedge^k \otimes E \;&\longrightarrow\; \bigwedge^{k+1} \otimes E \\ D(\eta\otimes\sigma) \;&=\; d\eta\otimes\sigma \;+\; (-1)^{deg(\eta)}\eta\wedge D\sigma. \end{aligned} \tag{126}$$

Suppose $\{\sigma_1, \ldots, \sigma_k\}$ is some local frame for E. Then we define the 1-forms θ_i^j by

$$\nabla\sigma_i \;=\; \theta_i^j \otimes \sigma_j. \tag{127}$$

The symbols θ_j^i are called the connection 1-forms.

Of course iterating the exterior derivative d gives zero: $d^2 = 0$. What happens when we iterate the covariant exterior derivative D?

$$\begin{aligned} DD(\eta\otimes\sigma) \;&=\; D\Big(d\eta\otimes\sigma + (-1)^{deg(\eta)}\eta\wedge D\sigma\Big) \\ &=\; dd\eta\otimes\sigma + (-1)^{deg(\eta)+1}d\eta\wedge D\sigma + d\eta\otimes\sigma + (-1)^{2deg(\eta)}\eta\wedge DD\sigma \\ &=\; \eta\wedge DD\sigma \end{aligned} \tag{128}$$

Therefore D^2, while not necessarily zero, is tensorial. In particular, if $\eta \in \bigwedge^0$ is any function f, then the computation above shows $D^2(f\sigma) = fD^2(\sigma)$. It is usual to denote this tensor by Θ, and call it the curvature tensor:

$$\begin{aligned} \Theta \,:\, \bigwedge^k \otimes E \;&\longrightarrow\; \bigwedge^{k+2} \otimes E \\ \Theta(\eta\otimes\sigma) \;&=\; D^2(\eta\otimes\sigma) \;=\; \eta\wedge D^2(\sigma). \end{aligned} \tag{129}$$

This is identical to the more familiar notion of the curvature of line bundles in Riemnnian geometry (see exercises). We can express this tensor in terms of the connection 1-forms by

$$\begin{aligned} D^2(\sigma_i) \;&=\; D\left(\theta_i^j\otimes\sigma_j\right) \\ &=\; d\theta_i^j\otimes\sigma_j \;-\; \theta_i^j\wedge\theta_j^k\otimes\sigma_k \\ &=\; \left(d\theta_i^j \;+\; \theta_i^s\wedge\theta_s^j\right)\sigma_j \end{aligned} \tag{130}$$

Therefore

$$\Theta_i^j \;=\; d\theta_i^j \;+\; \theta_i^s \wedge \theta_s^i \tag{131}$$

This is often abbreviated $\Theta = d\theta + \theta \wedge \theta$.

1.20 Holomorphic Vector Bundles and the Chern Connection

The theory of covariant exterior differentiation dovetails nicely with the holomorphic vector bundle theory, in the sense that holomorphic vector bundles automatically have a natural connection, the $\bar{\partial}$ operator, as we have already discussed. A connection $\nabla : E \to TM \otimes E$ is said to be *compatible with the holomorphic structure* of E provided $\nabla^{0,1} = \bar{\partial}$. Then if E has an hermitian inner product $\langle \cdot, \cdot \rangle$, we obtain the following uniqueness theorem:

Theorem 1.17 (Fundamental Theorem of Hermitian Holomorphic Vector Bundles) *If E is a holomorphic vector bundle over E with Hermitian inner product E, then there is a unique connection ∇ on E that satisfies*

- $\nabla : E \longrightarrow T^*M \otimes E$
- *(Leibniz rule)* $\nabla(f\sigma) = df \otimes \sigma + f\nabla\sigma$
- *(Compatibility with the Hermitian inner product)* $d\langle \sigma_1, \sigma_2 \rangle = \langle \nabla\sigma_1, \sigma_2 \rangle + \langle \sigma_1, \nabla\sigma_2 \rangle$
- *(Compatibility with the holomorphic structure)* $\nabla^{0,1}\sigma = \bar{\partial}\sigma$.

Before moving on, we should note that the curvature of such an Hermitian holomorphic vector bundle has a spacial bidegree decomposition: $\Theta_j^i \in \bigwedge^{1,1}$. Further, in a holomorphic frame Θ_j^i is purely imaginary and antihermitian in the sense that $\overline{\Theta_{i\bar{j}}} = -\Theta_{j\bar{i}}$.

This can be seen in two steps. Let $\{\sigma_i\}$ be a holomorphic frame, meaning $\bar{\partial}\sigma_i = 0$. Therefore $\nabla\sigma_i = \nabla^{1,0}\sigma_i$ implies $d\theta_j^i \in \bigwedge^{2,0} \oplus \bigwedge^{1,1}$ and $\theta \wedge \theta \in \bigwedge^{2,0}$ so that $\Theta \in \bigwedge^{2,0} \oplus \bigwedge^{1,1}$. Then we note

$$\begin{aligned} 0 \;&=\; d^2 \langle \sigma_i,\, \sigma_j \rangle \\ &=\; d\langle D\sigma_i,\, \sigma_j \rangle \;+\; d\langle \sigma_i,\, D\sigma_j \rangle \\ &=\; \langle \Theta(\sigma_i),\, \sigma_j \rangle \;-\; \langle D\sigma_i,\, D\sigma_j \rangle \;+\; \langle D\sigma_i,\, D\sigma_j \rangle \;+\; \langle \sigma_i,\, \Theta(\sigma_j) \rangle \\ &=\; \langle \Theta(\sigma_i),\, \sigma_j \rangle \;+\; \langle \sigma_i,\, \Theta(\sigma_j) \rangle \end{aligned} \tag{132}$$

so that $0 = \Theta_i^s h_{s\bar{j}} + \overline{\Theta_j^s} h_{i\bar{s}} = \Theta_i^s h_{s\bar{j}} + \overline{\Theta_j^s h_{s\bar{i}}}$.

Now we specialize to the case that (M, J) is a complex manifold, so $T_{1,0}M$ is a holomorphic vector bundle and has a $\bar{\partial}$ operator. Suppose (M, J, h) is a complex manifold with Hermitian inner product h. Then M has a natural connection, ∇_C, called the Chern connection—in general this is different from the Levi-Civita connection.

We would like to establish the conditions under which the $\nabla_C = \nabla_{L-C}$. That is, the conditions under which the holomprhic theory of curvature and the real (Riemnnian) theory coincide. This question leads us to the topic of Kähler geometry.

1.21 Problems for 2/18

45) Suppose ∇ is a connection on TM, and $\{\frac{\partial}{\partial x^i}\}$ is a coordinate frame. Show that the connection 1-forms are just the Christoffel symbols: $\theta^k_j = \Gamma^k_{ij} dx^i$.

46) In the Riemannian setting where $E = TM$, show that Θ^l_k is indeed the Riemannian curvature operator.

47) Consider the 2-sphere, with metric given by $g_{ij} = (1 + r^2)^{-2}\delta_{ij}$. Determine the connection 1-forms θ^i_j and the curvature tensor Θ^i_j.

48) In the real setting, what can you say about the traced curvature 2-form $\rho = \Theta^i_i$?

49) Prove the *second Bianchi identity*: $d\Theta + \theta \wedge \Theta + \Theta \wedge \theta = 0$. Hint: start by showing $D(\Theta(\sigma)) = \Theta(D(\sigma))$, and work out what this means in terms of the θ-symbols.

50) Starting with $dh_{i\bar{j}} = d\langle e_i, e_j\rangle = \theta^s_i h_{s\bar{j}} + \overline{\theta^s_j} h_{s\bar{j}}$ and pairing off by bi-degree, show that $\theta^i_j = h^{i\bar{s}}\partial h_{j\bar{s}}$ and that $\Theta^i_j = \bar{\partial}\left(h^{i\bar{s}}\partial h_{j\bar{s}}\right)$

1.22 Chern Classes and Kähler Manifolds

1.22.1 The Ricci form and the First Chern Class

We shall not attempt to develop the theory of characteristic classes or even Chern classes of complex vector bundles, but shall content ourselves with describing the first Chern class of a holomorphic vector bundle.

Given an Hermitian holomorphic vector bundle $(E, \bar{\partial})$ over a holomorphic manifold M^n of complex dimension n, we have the curvature operator

$$\begin{aligned} \Theta &\in \bigwedge^2 \otimes E^* \otimes E \\ \Theta &= \Theta^i_j \otimes e^j \otimes e_i. \end{aligned} \tag{133}$$

where $\{e_i\}$ is any local holomorohic basis for E and $\{e^i\}$ is the corresponding dual basis for E^*. If $h = h_{i\bar{j}} e^i \otimes \overline{e^j}$ is the Hermitian form, then we have the array of 2-forms Θ^i_j given by

$$\Theta^i_j = \bar{\partial}\left(h^{i\bar{s}}\partial h_{j\bar{s}}\right) \tag{134}$$

(see previous exercises). But we can trace Θ in its $E^* \otimes E$ factor to obtain (a complex multiple of) the Hermitian holomorphic vector bundle's Ricci 2-form, which we define to be

$$\rho = \sqrt{-1}\Theta^i_i \in \bigwedge^2 \tag{135}$$

Recall the *Jacobi formula* for matrices: if $A = (A_{ij})$ is a matrix dependent on some variable x, then $\frac{d}{dx} \log A = A^{ji} \frac{dA_{ij}}{dx}$. Using this, we obtain

$$\rho \;=\; \sqrt{-1}\bar{\partial}\left(h^{i\bar{s}}\partial h_{i\bar{s}}\right) \;=\; -\sqrt{-1}\partial\bar{\partial} \log \det(h), \tag{136}$$

which is clearly a $(1,1)$-form. We shall show below that it is a *real* (1,1)-form, but for now we will take this for granted. Because $\bar{\partial}\partial(f) = d\partial(f)$ (as a result of $\partial^2 = 0$, we see that ρ is locally exact, and therefore it is globally closed. Therefore $[\rho]$ represents a deRham cohomology class.

However, it is not yet clear that this class represents anything fundamental, because within the definition of ρ is a choice of Hermitian form h on E. So suppose h and k are two Hermitian forms on E. Then we can connect them through a path:

$$h_t \;\triangleq\; (1-t)h \,+\, tk. \tag{137}$$

Note that h_t is an Hermitian metric for all $t \in [0,1]$, and that $h_0 = h$ and $h_1 = k$. For each h_t we have a new ρ_t, but we see

$$\begin{aligned} \frac{d}{dt}\rho \;&=\; -\sqrt{-1}\partial\bar{\partial}\,\frac{d}{dt}\log\det\, h_t \;=\; -\sqrt{-1}\partial\bar{\partial}\,\left((h_t)^{i\bar{j}}(k_{i\bar{j}} - h_{i\bar{j}})\right) \\ &=\; d\left(-\sqrt{-1}\bar{\partial}\, Tr_{h_t}\,(k-h)\right) \end{aligned} \tag{138}$$

But $Tr_{h_t}(k-h)$ is a globally defined function. Therefore $\frac{d}{dt}\rho_t$ is an exact 2-form, which means that the class $[\rho_t]$ is actually invariant. The class $[\rho]$ is called the *first Chern class* of $(E, \bar{\partial})$, and denoted $c_1(E)$. This gives the following remarkable theorem:

Theorem 1.18 (The First Chern Class) *Assume $(E, \bar{\partial})$ is a holomorphic vector bundle. Given any Hermitian form h on E, the resulting Ricci 2-form ρ is real and closed. The resulting deRham class $[\rho] \in H^2_{dR}(M^n, \mathbb{R})$ is invariant under choice of Hermitian form h, and is therefore a constant of the holomorphic structure $\bar{\partial}$ of the bundle E.*

1.22.2 Hermitian, Metric, and Kähler forms on Complex Manifolds

The theory of Chern curvature obviously exists when we choose $E = T'_{\mathbb{C}}M$, the holomorphic tangent bundle. The resulting first Chern class $c_1(TM) = c_1(M)$ is referred to as the first Chern class of the manifold itself.

Let $h = h_{i\bar{j}} dz^i \otimes d\overline{z^j}$ be an Hermitian metric on the complex n-manifold M^n. We can separate h into its symmetric and anti-symmetric parts:

$$\begin{aligned} h \;&=\; h_{i\jmath} dz^i \otimes d\bar{z}^j \\ &=\; \frac{1}{2} h_{i\jmath} dz^i \odot \bar{z}^j \;+\; \frac{1}{2} h_{i\jmath} dz^i \wedge d\bar{z}^j \end{aligned} \tag{139}$$

We make the definitions

$$\begin{aligned} g &= \frac{1}{2} h_{i\bar{j}} dz^i \odot \bar{z}^j \\ \omega &= \frac{\sqrt{-1}}{2} h_{i\bar{j}} dz^i \odot \bar{z}^j \end{aligned} \tag{140}$$

so that $h = g - \sqrt{-1}\omega$. Notice that, since $\overline{h_{i\bar{j}}} = h_{j\bar{\imath}}$, we have

$$\begin{aligned} \bar{g} &= \frac{1}{2} \overline{h_{i\bar{j}} dz^i \odot d\bar{z}^j} \;=\; \frac{1}{2} h_{j\bar{\imath}} d\bar{z}^i \odot dz^j \\ &= \frac{1}{2} h_{j\bar{\imath}} dz^j \odot d\bar{z}^i \;=\; g. \end{aligned} \tag{141}$$

Likewise we can compute $\overline{\omega} = \omega$, so ω is also real. The real symmetric tensor g is called the Riemannian metric associated to h, and the real antisymmetric 2-tensor ω is called the Kähler form associated to h. We can prove that $\omega(\cdot, \cdot) = g(J\cdot, \cdot)$ (see exercises).

Further, it is a simple matter of algebra to prove that

$$dVol_g \;=\; \frac{1}{n!} \omega \wedge \cdots \wedge \omega \tag{142}$$

(n-fold wedge product), and that

$$*\omega \;=\; \frac{1}{(n-1)!} \omega \wedge \cdots \wedge \omega \tag{143}$$

($(n-1)$-form wedge product), so that $|\omega|^2 = *\,(\omega \wedge *\omega) = n$.

1.22.3 The Kähler condition and Initial Considerations

Every complex manifold has at least one Hermitian inner product; this can be obtained, for instance, by using creating locally defined Hermitian forms, and gluing them together with a partition of unity. But this is too "flexible" to be of further use.

If the Kähler 2-form ω satisfies the *Kähler condition* that

$$d\omega \;=\; 0, \tag{144}$$

then we say that (M^n, J, ω) is a Kähler manifold, and that the associated metric g is a Kähler metric.

This is actually a restrictive condition, and not every complex manifold admits a Kähler metric. For instance, if M^n is compact, note that

$$\mathrm{Vol}(M^n) \;=\; \frac{1}{n!} \int_{M^n} \omega \wedge \cdots \wedge \omega. \tag{145}$$

However, if $\omega = d\eta$ is actually exact, then $\omega \wedge \cdots \wedge \omega = d(\eta \wedge \omega \wedge \cdots \wedge \omega)$ is also exact, meaning $\int \omega \wedge \cdots \wedge \omega = 0$, an impossibility. Therefore ω is closed but not exact, and therefore defines a non-zero cohomology class $[\omega]$, called the *Kähler class* of the manifold.

1.22.4 Various Forms of the Kähler Condition, and the Equivalence of Real and Complex Geometry on Kähler manifolds

Setting $\omega = \omega_{i\bar{j}} dz^i \wedge d\bar{z}^j$ where of course $\omega_{i\bar{j}} = \sqrt{-1} h_{i\bar{j}}$, the Kähler condition is

$$0 \;=\; d\omega \;=\; \frac{d\omega_{i\bar{j}}}{dz^k} dz^k \wedge dz^i \wedge d\bar{z}^j + \frac{d\omega_{i\bar{j}}}{d\bar{z}^k} d\bar{z}^k \wedge dz^i \wedge d\bar{z}^j. \tag{146}$$

for this to be zero, it is necessary and sufficient that

$$\frac{d\omega_{i\bar{j}}}{dz^k} \;=\; \frac{d\omega_{k\bar{j}}}{dz^i} \tag{147}$$

for all i, j, k. But this has an expression in terms of Christoffel symbols: because $\omega_{i\bar{j}} = \sqrt{-1} h_{i\bar{j}}$, we have

$$\begin{aligned} \frac{d}{dz^k} h_{i\bar{j}} \;&=\; \left\langle \nabla_{\frac{d}{dz^k}} \frac{d}{dz^i}, \, \frac{d}{dz^j} \right\rangle \;+\; \left\langle \frac{d}{dz^i}, \, \nabla_{\overline{d/dz^k}} \frac{d}{dz^j} \right\rangle \\ &=\; \Gamma_{ki}^s h_{s\bar{j}} \\ \frac{d}{dz^i} h_{k\bar{j}} \;&=\; \Gamma_{ik}^s h_{s\bar{j}}. \end{aligned} \tag{148}$$

Therefore the Kähler condition is equivalent to $\Gamma_{ik}^t = \Gamma_{ki}^t$. However this is precisely the torsion-free condition of Riemannian geometry.

Theorem 1.19 *The Chern connection on an Hermitian complex manifold is the Levi-Civita connection if and only if the metric is Kähler.*

Theorem 1.20 *An Hermitian complex manifold (M^n, J, h) is Kähler if and only if J is covariant-constant with respect to the Chern connection: $\nabla_Y(JX) = J\nabla_Y X$.*

Proof. Exercise. □

1.23 Problems for 2/23

51) Prove that $\overline{\omega} = \omega$, so that ω is a real 2-form.

52) Defining g, ω from h as above, prove that $g(J\cdot, J\cdot) = g(\cdot, \cdot)$ and $\omega(J\cdot, J\cdot) = \omega(\cdot, \cdot)$.

53) Defining g, ω from h as above, prove that $\omega(\cdot, \cdot) = g(J\cdot, \cdot)$.

54) Prove the converse of (52) and (53), namely if g is J-invariant, then setting $\omega(\cdot, \cdot) = g(\cdot, \cdot)$, show that ω is a J-invariant, antisymmetric 2-form, and that $h = g - \sqrt{-1}\omega$ is an Hermitian metric.

55) If (M^n, J, g) is a complex manifold with metric g, show that this is a Kähler manifold provided the two compatibility conditions $g(J\cdot, J\cdot) = g(\cdot, \cdot)$ and $dg(J\cdot, \cdot) = 0$ both hold.

56) Verify the assertion before (147). In particular, show that $\frac{d\omega_{i\bar{j}}}{dz^k} = \frac{d\omega_{k\bar{j}}}{dz^k}$ is equivalent to $\frac{d\omega_{i\bar{j}}}{d\bar{z}^k} = \frac{d\omega_{i\bar{k}}}{d\bar{z}^j}$

57) Prove Theorem 1.20.

1.23.1 The Hodge Stars $*$ and $\bar{*}$, and the Bidegree Decomposition

1.23.2 The $\partial\bar{\partial}$-operator

1.23.3 Signed Forms. Pluriharmonic and Plurisubharmonic functions

1.23.4 Signed Cohomology Classes

1.23.5 The Calabi Conjecture for Kähler-Einstein Metrics

1.24 Exercises for 2/25