Lectures on the Theory of Collapsing of Riemannian Manifolds

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These lectures are entirely expository; the results and methods are due to others. Appropriate credit is indicated throughout the notes.

These lecture notes were written to accompany my teaching of Math 683 at Stony Brook University in the spring of 2010. Each "chapter" represents one day's lecture material. These are rough lectures notes, unedited, and compiled on a day-to-day basis, so there are sure to be typos, inaccuracies, and other errata. Any mistakes are mine alone.

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Chapter 1

Examples of Collapsing Manifolds

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1.1 Collapsing

We are mainly interested in the phenomenon of sequences of manifolds with injectivity radii limiting to zero, while sectional curvatures remain bounded.

Any compact Riemannian manifold can be said to converge to a point by multiplying its metric by a constant δ^2 and letting $\delta \to 0$. What is meant by "converge to a point" will be made precise later, but note that such a manifold's volume and diameter do converge to zero. This kind of process in rather trivial. Note that only in the flat case does this produce a family with bounded curvature.

However, many Riemannian manifolds admit some collapsing process that leaves curvature bounded. This is "geometric collapse," or more properly *collapse with bounded curvature.*

1.2 Example: $\mathbb{S}^3$

The first example (historically) is Berger's collapsing 3-sphere. We first describe the classical Hopf fibration. The unit 3-sphere can be defined as the set of points $(z, w) \in \mathbb{C}^2$ with $|z|^2 + |w|^2 = 1$. The 1-sphere $\mathbb{S}^1 = \{e^{i\theta} \in \mathbb{C} \mid \theta \in \mathbb{R}\}$ acts on $\mathbb{S}^3$ by multiplication: if

$\phi = e^{i\theta} \in \mathbb{S}^1$ we define

$$\phi.(z, w) \;=\; \left(e^{i\theta} z,\, e^{i\theta} W\right).$$

The orbit of a point under this $\mathbb{S}^1$ action are called Hopf circles. This generates a foliation of $\mathbb{S}^3$ by $\mathbb{S}^1$, that is actually a fiber bundle (actually a principle bundle).

There is also a simple map $\mathbb{S}^3 \to \mathbb{C}^* \approx \mathbb{S}^2$, called the Hopf map, given by

$$(z,\, w) \mapsto\; z w^{-1}.$$

It is easy to show that if (z, w) and $(\tilde{z}, \tilde{w})$ map to the same point in $\mathbb{S}^2$ then $(z, w) = c(\tilde{z}, \tilde{w})$ for some constant c, with, necessarily, $|c| = 1$. Therefore $c \in \mathbb{S}^1$, and we see that the fibers of this submersion are precisely the Hopf circles. The O'Niell formulas show that $\mathbb{S}^2$ is a half-radius sphere, of constant sectional curvature 4.

It is possible to see this map in a more "active" setting. Topologically $\mathbb{S}^3$ is just the Lie group $SU(2)$. Let X, Y, Z be a basis of its Lie algebra $\mathfrak{g} = \mathfrak{su}(2)$ with brackets

$$[X,\, Y] \;=\; 2Z \quad [Y,\, Z] \;=\; 2X \quad [Z,\, X] \;=\; 2Y,$$

and dual basis $\eta, \mu, \zeta \in \mathfrak{g}^*$. Let $g = \eta \otimes \eta \,+\, \mu \otimes \mu \,+\, \zeta \otimes \zeta$ be the standard bi-invariant metric on $SU(2)$; in fact this is the round metric on $\mathbb{S}^3$:

$$\begin{aligned} R(X,Y)Z \;&=\; -\frac{1}{4}[[X,Y],Z] \;=\; 0 \\ R(X,Y)Y \;&=\; -\frac{1}{4}[[X,Y],Y] \;=\; X. \end{aligned}$$

Now let $g_\delta \;=\; \delta^2 \eta \otimes \eta \,+\, \mu \otimes \mu \,+\, \zeta \otimes \zeta$ be another metric; g_δ is left-invariant but not bi-invariant. One verifies that sectional curvatures are bounded, but that the injectivity radius at each point is $|\delta|$, as given by the geodesic with tangent vector X. The limiting object is $\mathbb{S}^2$ with sectional curvature 4.

1.3 Example: Free effective torus actions

The 3-sphere example can be generalized. Berger's collapse is just the scaling of the metric along the orbits of the circle-action while leaving the metric unchanged on the perpendicular distribution. Now suppose a torus T^k acts freely (isotropy groups are trivial) and isometrically on a Riemannian manifold M^{n+k}. Now M supports an integrable tangential distribution (of dimension k) and a perpendicular distribution (of dimension n). The metric can be likewise decomposed: letting $T_p \subset T_pM$ and $P_p \subset T_pM$ indicate the tangential and perpendicular distributions, respectively, we can write $g \;=\; g_T \,+\, g_P$.

Not let $g_\delta \;=\; \delta^2 g_T + g_P$. Pick a point $p \in M$; we will estimate the sectional curvatures at p. Let $N^n \subset M^{n+k}$ be a transverse submanifold, defined in a neighborhood of p, that contains p. Let $y^1, \ldots, y^n$ be coordinates on N with $p = (0, \ldots, 0)$. Let $\tilde{x}^1, \ldots, \tilde{x}^k$ be coordinates on T^k with the identity e having coordinates $x^i = 0$. The freeness of the actions allows these coordinate functions to push forward to functions $x^1, \ldots, x^k$ locally near p, where $x^1 = \cdots = x^k = 0$ on the transversal N. Finally extend the functions $y^1, \ldots, y^n$ to a neighborhood of p by projection along the fibers onto N. This gives a coordinate system $\{x^1, \ldots, x^n, y^1, \ldots, y^n\}$ in a neighborhood of p.

The coordinate fields $\frac{\partial}{\partial x^i}$ are tangent to the fibers, although the fields $\frac{\partial}{\partial y^i}$ are not. Write $\frac{\partial}{\partial y^i} = X_i + V_i$ where the fields X_i are parallel to the fibers and the fields V_i are perpendicular to the fibers.

The original metric has the form

$$g = \begin{pmatrix} A & B \\ B & C+D \end{pmatrix},$$

where $A_{ij} = \langle \frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j} \rangle$, $B_{ij} = \langle \frac{\partial}{\partial x^i}, X_j \rangle$, $C_{ij} = \langle X_i, X_j \rangle$, and $D_{ij} = \langle V_i, V_j \rangle$. Note that these matrices are functions of the coordinates y^i only, since the torus action is isometric. The new metrics have the form

$$g_\delta(x, y) = \begin{pmatrix} \delta^2 A & \delta^2 B \\ \delta^2 B & \delta^2 C + D \end{pmatrix}.$$

This metric is singular and it is not clear that sectional curvature remains bounded. Now make a change of coordinates: $u^i = \delta x^i$, and let $\tilde{A}_{ij} = \langle \frac{\partial}{\partial u^i}, \frac{\partial}{\partial u^j} \rangle$ and $\tilde{B}_{ij} = \langle \frac{\partial}{\partial u^i}, X_i \rangle$. Then in the new coordinates we have

$$g_\delta(u, y) = \begin{pmatrix} \delta^2 \tilde{A} & \delta^2 \tilde{B} \\ \delta^2 \tilde{B} & \delta^2 C + D \end{pmatrix} = \begin{pmatrix} A & \delta B \\ \delta B & \delta^2 C + D \end{pmatrix}$$

$$\lim_{\delta \to 0} g_\delta = \begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix},$$

a generalized warped product metric. It is clear now that the g_δ have bounded curvature. It is also possible to prove that injectivity radii converge to 0, so we indeed have a prototype for collapse with bounded curvature.

Theorem of Cheeger-Gromov: This example is in essence the only kind of collapse with bounded curvature, at least as observed on the scale of the injectivity radius.

1.4 Example: Nilmanifolds

Nilmanifolds provide the prototype for collapse with bounded curvature. Note that tori are Lie groups with Abelian algebras.

Let G be a finite dimensional Lie group with Lie algebra $(\mathfrak{g}, [,])$. Its *descending central series* is defined inductively by $\mathfrak{g}^0 = \mathfrak{g}$ and $\mathfrak{g}^i = [\mathfrak{g}^{i-1}, \mathfrak{g}]$. If $\mathfrak{g}^k = 0$ for some k, then $\mathfrak{g}$ is called a *nilpotent Lie algebra* and g a *nilpotent Lie group*. Note the possible conflict with the use of this term in the group-theoretic setting. In fact there is no conflict. The Lie algebra operation $[,]$ and the group's commutator operation (also denoted $[,]$) are related by the formula

$$[X_1, X_2] \;=\; \frac{1}{2}\frac{d^2}{dt^2}[\exp(tX_1), \exp(tX_2)].$$

This can be exploited easily to show that a Lie group that is nilpotent in the group-theoretic sense is nilpotent in the Lie algebra sense. The converse is slightly more difficult to see, but also true.

Any (finite-dimensional) nilpotent Lie group N is isomorphic to a group of $n \times n$ upper-triangular matrices with 1's along the diagonal; it's Lie algebra is a Lie algebra of strictly upper triangular matrices (ie. with 0's along the diagonal). Given $q > 0$ let g_q be the norm

$$|A|^2 \;=\; \sum_{ij} (a_{ij})^2 \, q^{2(i-j)}.$$

This gives rise to a left-invariant metric that is bi-invariant only when $q = 1$. Alternatively we can let $\mathfrak{g}_k$ be the vector space of matrices a_{ij} where $a_{ij} = 0$ unless $i - j = k$, and let g_q be the metric so that $g_q|_{\mathfrak{g}^k} = q^{2k} g$. Because $[\mathfrak{g}_k, \mathfrak{g}_l] \subset \mathfrak{g}_{k+l}$ (a simple consequence of the Jacobi identity), we get the estimate

$$|[X,Y]|_q^2 \;\leq\; |XY - YX|_q^2 \;\leq\; 2(n-2)|X|_q^2 |Y|_q^2,$$

which we use to derive the estimates

$$\begin{aligned} |\nabla_X Y|_q \;&\leq\; 3\sqrt{2(n-2)}|X|_q^2|Y|_q^2 \\ |\operatorname{Rm}(X,Y)Z|_q \;&\leq\; 42(n-2)|X|_q|Y|_q|Z|_q \end{aligned}$$

(hint for (1.1): use the Koszul formula). This implies that sectional curvatures are bounded independently of q.

Let N denote the Lie group of upper triangular matrices under consideration, and let $\Gamma \subset N$ be a cocompact subgroup; for instance the group of upper triangular matrices with integral entries and 1's along the diagonal. The metric is invariant under left translation by Γ so the metric descends to the quotient $M = \Gamma$
N. As $q \to 0$ the diameter of M goes to zero while its sectional curvature remains bounded. Thus M is an example of an "almost-flat manifold", one which supports a sequence of metrics with $\operatorname{diam}(M)^2 \cdot \max|\sec(M)| \searrow 0$. We shall prove later that there is no flat metric on M.

Gromov's theorem states that this is in fact the only way for a manifold to collapse to a point with bounded curvature.

Consider the Heisenberg group, the group of upper triangular 4×4 matrices with 1's along the diagonal. It is 3-dimensional, its natural bi-invariant metric has both positive and negative curvatures, and its Riemann tensor is not parallel. The left quotient by the integer subgroup gives a twisted circle-bundle over the torus. This is the (compact) prototype for Thurston's nilgeometry.

1.5 Example: Solvmanifolds

This previous example can be extended to the solvegeometry. Let G be a solvable Lie group, given by the upper triangular $n \times n$ matrices, with Lie algebra $\mathfrak{g}$. Define a sequence of normal subgroups by $G^0 = G$, $G^1 = [G, G]$ and $G^k = [G^1, G^{k-1}]$, and put on a metric g_q so that $v \in G^i$ and v is perpendicular to G^{i-1} gives $|v|^2 = q^{2i}$.

Let $\Gamma \subset G$ be a cocompact discrete subgroup, for instance the integer subgroup. Now sending $q \to 0$ we get collapse to a manifold $\Gamma \backslash (G/G^1)$, which is isometric to the torus T^n. The "collapsed" directions constitute a fibration by nilmanifolds, each isomorphic to $(G^1 \cap \Gamma) \backslash G$. Indeed this produces a fiber bundle

$$(G_1 \cap \Gamma) \backslash G_1 \to \Gamma \backslash G \to \Gamma \backslash G / G_1.$$

This is an example of Fukaya's theorem, than any collapse to a lower dimensional Riemannian manifold produces a fiber bundle, where the fibers are nilmanifolds, along which the collapse occurs.

Chapter 2

Topology and Convergence in the Space of Metric Spaces

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2.1 The Hausdorff distance

2.1.1 Basic Properties

Given a bounded metric space X, the set of closed sets of X supports a metric, the Hausdorff metric. Whether X is bounded or not, there is a compact, locally compact topology on the space of closed sets. If $A, B \subset X$ are closed sets, define their *Hausdorff distance* $d_H(A, B)$ to be the number

$$\inf \{ \, r \mid B \, \text{is in the} \, r - \text{neighborhood of} \, A \, \text{and} \, A \, \text{is in the} \, r - \text{neighborhood of} \, B \, \}.$$

We can say this more precisely as follows. We say B is r-close to A (or B is in the r-neighborhood of A) if

$$B \; \subset \; \bigcup_{x \in A} B(x, r).$$

Then the Hausdorff distance is the infimum of all r such that B is r-close to A and A is r-close to B. There is still another equivalent definition. Given a point $p \in X$ and a closed set $A \subset X$, define

$$d(p, A) \; = \; \inf_{y \in A} \, \text{dist}(p, y).$$

Then the Hausdorff distance is

$$d_H(A,B) \;=\; \max\left\{\sup_{x\in A} d(x,B),\; \sup_{y\in B} d(y,A)\right\}$$

That is, $d_H(A,B)$ is the farthest distance any point of B is from the set A, or the farthest any point of A is from B, whichever is greater.

Theorem 2.1.1 *If X is a bounded metric space, the set of closed sets of X is itself a metric space with the Hausdorff metric.*

Pf We verify the metric space axioms. First, the symmetry of d_H is clear by definition. Second, d_H satisfies the triangle inequality because if C is in the r-neighborhood of B and B is in the s-neighborhood of A, then C is in the $(r+s)$-neighborhood of A. Likewise A is in the $(r+s)$-neighborhood of C. Thus $d(A,C) \leq d(A,B) + d(B,C)$. Finally $d_H(A,B) = 0$ implies $A \subset \overline{B} = B$, because if B is in every r-neighborhood of A then every point of A is a limit point of B. Likewise $B \subset \overline{A} = A$. □

If X is not bounded, the metric space axioms continue to hold, but $d_H(A,B)$ could well be infinity.

2.1.2 Compactness

Denote the closed subset of X by $\mathfrak{C}(X)$ (or just $\mathfrak{C}$ for short). Given a closed set A and a number r, let $\mathfrak{B}(A,r)$ be the set of all $D \in \mathfrak{C}$ with $d_H(B,A) < r$. Since d_H is a metric on $\mathfrak{C}$, we know that the balls $\mathfrak{B}(A,r)$ are open, and form a neighborhood base.

Obviously the balls with rational radius also form a base, so the induces topology on $\mathfrak{C}$ is *first countable*. All metric spaces are Hausdorff, so $(\mathfrak{C}, d_H)$ is Hausdorff. One can state this directly: since distinct closed sets are separated by a finite distance, say ϵ, so the balls of radius, say, $\epsilon/4$ around each is disjoint.

If X is noncompact, then the topology associated to the Hausdorff distance is neither compact nor even locally compact. To see the local noncompactness, simply pick a sequence $x_i \in X$ that has no convergent subsequence, and define the closed sets X_i to be $X_i = \{x_j\}_{j=1}^{i}$. Given any neighborhood $\mathfrak{N}$ of $X_\infty = \{x_j\}_{j=1}^{\infty}$, each $X_i \in \mathfrak{N}$.

If X is noncompact, $(\mathfrak{C}(X), d_H)$ is not even locally compact. For instance if the base space X is nondiscrete (it has the property that, given any point $x \in X$ and any number $\epsilon > 0$, there is a point $y \in X$ with $d(x,y) < \epsilon$), then it is not locally compact. As an example, we will will show that $\mathbb{R}$ is not locally compact. Let $A = [0,\infty)$ be the half-line, and consider its r-neighborhood $B(A,r)$ (wlog assume $r < \frac{1}{2}$). Define the A_i inductively by setting $A_0 = A$ and $A_i = A_{i-1} - (i, i + r/2)$. We have $d_H(A_i, A_j) = r/2$ for any $i \neq j$, so there are no Cauchy subsequences, and therefore no convergent subsequences.

In fact, the metric topology on $(\mathfrak{C}(\mathbb{R}), d_H)$ is not even locally paracompact. There exist closed sets A such that every neighborhood of A contains an uncountable discrete subset.

In sharp contrast we have the following theorem.

Theorem 2.1.2 *If X is compact, then $(\mathfrak{C}(X), d_H)$ is compact.*

Pf

Let A_i be a sequence of open sets. Each A_i, has a $\frac{1}{j}$-net consisting of $< N_j \in \mathbb{N}$ elements (an ϵ-net is a maximal discrete ϵ-separated subset; the compactness of X guarantees the existence of the number N_j). Let $A_i^k \subset A_i$ be the union of the $\frac{1}{j}$-nets in A_i for $1 \leq j \leq k$; note that the cardinality of A_i^k is at most $N_1 + \cdots + N_j$.

Fixing k, some subsequence $A_{i_k}^k$ converges in the Hausdorff topology, to a some discrete set A^k. Since $A_{i_k}^k$ is $\frac{1}{k}$-close to A_{i_k}, we have that, for large i_k, A_{i_k} is $\frac{3}{k}$-close to A^k. We can require that $A_{i_k}^k$ is a subsequence of $A_{i_{k+1}}^{k+1}$, which means $A^k \subset A^{k+1}$. Since A^k is ϵ-close to $A_{i_k}^k$ for large i_k, and $A_{i_{k+1}}^k$ is $\frac{1}{k}$-close to $A_{i_{k+j}}^{k+j}$ which is ϵ-close to A^{k+j}, we have that A^k is $(\frac{1}{k} + 2\epsilon)$-close to A^{k_j}, any $\epsilon > 0$ so that A^k is $\frac{1}{k}$-close to A^{k+j}.

The diagonal subsequence $A_{k_k}^k$ converges to some set A^∞, in which each A^k is $\frac{1}{k}$-dense. Consider the sequence A_{k_k}. Since $A_{k_k}^k$ is $\frac{1}{k}$-dense in A_{k_k} and is also $\frac{1}{k}$-close to A^∞, this means A_{k_k} is $\frac{1}{k}$-close to A^∞. This implies A_{k_k} converges to $\overline{A^\infty}$. □

A topology does exist on $\mathfrak{C}(X)$ that is both locally compact and compact, regardless of the compactness of X. Let a base for this topology be set of the form $N_{K,\epsilon}(A)$, where $K \subset X$ is compact, $A \subset X$ is closed, and $\epsilon > 0$, where we define

$$N_{K,\epsilon}(A) \;=\; \{\, B \in \mathfrak{C}(X) \mid d_H(A \cap K, B \cap K) \;<\; \epsilon \,\}.$$

This topology on $\mathfrak{C}(X)$ is called the *pointed Hausdorff topology.* If X is compact, it is the metric topology. If X is noncompact, this topology is not induced by any metric.

2.2 The Gromov-Hausdorff distance

The Gromov-Hausdorff distance was invented by Gromov for the purpose of making precise the notions of “closeness” and “convergence.” Recall that his “Almost Flat Manifold” theorem states that a compact bounded-curvature manifold that is “close” to being a point has a finite normal cover that is “close” to being a nilmanifold. The idea behind the Gromov-Hausdorff distance is not difficult; here is what Gromov himself has to say:

- “Either you have no inkling of an idea or, once you have understood it, the very idea appears so embarrassingly obvious that you feel reluctant to say it aloud... ”

- "I knew [of] it [the Gromov-Hausdorff metric] for a long time, but it just seemed too trivial to write. Sometimes you just have to say it."[1]

The Gromov-Hausdorff distance significantly extends the idea of the Hausdorff distance (and is not equivalent to it). Given two closed subsets A and B of any metric space (not necessarily subsets of the same space), we define

$$d_{GH}(A,B) \;=\; \inf_{f,g} d_H(f_{A\to X}(A),\, g_{B\to X}(B))$$

where the notation $f_{A\to X}$ (resp. $g_{B\to X}$) denotes an isometric embedding of A into some metric space X (resp. isometric embeddings of B into X) and the infimum is taken over all possible such embeddings.

In general the topology associated to the Gromov-Hausdorff distance is neither locally compact nor locally paracompact. To redress this we define the *pointed Gromov-Hausdorff topology*. This is a topology on the set of pointed sets (defined to be pairs (A,p) where A is a closed subset of a metric space and $p \in A$). A local base for this topology are the sets of the form $N_{K,\epsilon}(A)$ (where A is closed, $K \subset A$ is compact and $p \in K$, and $\epsilon > 0$); we define $N_{K,\epsilon}(A)$ to be the set of pointed closed sets (B,q) so that there exists a compact subset $J \subset B$, $q \in J$, and so that there are isometric embeddings $f : A \cap K \to X$ and $g : B \cap J \to X$ into some space X so that $f(p) = g(q)$ and the Hausdorff distance satisfies $d_H(f(A\cap K),\, g(B\cap J)) < \epsilon$.

This topology is locally compact and compact. If the Gromov-Hausdorff topology is restricted to compact closed sets, the Gromov-Hausdorff topology and the pointed Gromov-Hausdorff topology coincide.

2.3 The Lipschitz, $C^{k,\alpha}$, and $L^{p,k}$ topologies

The Gromov-Hausdorff topology is not suitable for questions of differentiability or even topology, since Gromov-Hausorff limits can jump differentiable structures, topologies, and even dimensions. For example a sequence of tori can converge to a round sphere, or to a circle or to a point.

Thus the Gromov-Hausdorff topology is completely inadequate when studying Riemannian structures (curvature, etc), and we have to find something sharper. Let $f : M \to N$ be a map between metric spaces. Define the dilation of f to be

$$dil(f) \;=\; \sup_{p,q\in M} \left\{ \frac{\mathrm{dist}_N(f(p),f(q))}{\mathrm{dist}_M(p,q)} \right\}.$$

We allow $dil(f)$ to take values in $[0,\infty]$. We define the *Lipschitz distance* between compact

[1]Taken from Cheeger's lecture 'Mikhail Gromov: How Does He Do It?'.

homeomorphic metric spaces M, N by

$$Lip(M,N) \;=\; \inf_{\substack{f:M\to N,\\ f\,homeo}} |\log(dil(f))| \;+\; |\log(dil(f^{-1}))|.$$

One easily verifies that this is a metric (up to equivalence of isometric metric spaces). If M is compact then the induced topology is locally compact. If M is noncompact, one can define a "local Lipschitz topology," meaning convergence occurs iff it occurs when restricted to compact subsets of the original metric spaces M, N. The convergence is essentially of Lipshitz type: for instance the graphs of $\frac{1}{n}\sin(n\pi t)$ over the unit interval for $n \in \mathbb{Z}$ converge to the unit interval. If one includes Riemannian metrics of type $C^{0,1}$, then the space of Riemannian metrics on a compact manifold M is locally compact and complete in the Lipschitz topology; this can be seen by examining the sequence of metrics on a chart in M diffeomorphic to a Euclidean ball and applying the Arzela-Ascoli theorem.

It is possible to further refine the Lipschitz topology in the category of Riemannian manifolds. Given a sequence of Riemannian manifolds (M_i, g_i), one says that they converge to (M, g) in the $C^{k,\alpha}$- or $L^{k,p}$-topology if there are homeomorphisms $f : M \to M_i$ such that the following holds: Given any coordinate chart $U \subset M$ with coordinates $\{x^1, \ldots, x^n\}$, with pullback metrics $g_{i,jk}dx^j \otimes dx^k$, the functions $g_{i,jk}$ converge to g_{jk} in the $C^{k,\alpha}$- or $L^{p,k}$-sense.

Chapter 3

The Gromov-Hausdorff Topology

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Gromov-Hausdorff distance and the Gromov-Hausdorff topology are central to these lectures.

3.1 Equivalent formulations of the Gromov-Hausdorff distance

Proposition 3.1.1 *The Gromov-Hausdorff distance $d_{GH}(X, Y)$ is the infimum of the Hausdorff distances between X and Y taken among all metrics on $X \coprod Y$ that restrict to the given metric on X and on Y.*

Pf

Define $\tilde{d}_{GH} = \inf\{d_H(X, Y)\}$, where the infimum is taken over metrics on $Z = X \coprod Y$ that restrict to the given metrics on X and Y. Since d_{GH} is an infimum taken over a larger set,

$$d_{GH} \le \tilde{d}_{GH}.$$

Now consider a metric on some ambient space Z that restricts to the given metrics on X and Y. Put $\alpha = d_H(X, Y)$. Define a function $\bar{d} : X \coprod Y \times X \coprod Y \to \mathbb{R}^{\ge 0}$ as follows. If $x_1, x_2 \in X$ and $y_1, y_e \in Y$, set $\bar{d}(x_1, x_1) = d(x_1, x_2)$, $\bar{d}(y_1, y_2) = d(y_1, y_2)$, $\tilde{d}(x_1, y_1) = d(x_1, y_1)$ if $d(x_1, y_1) \ge \alpha/2$, and $\bar{d}(x_1, y_1) = \alpha/2$ if $d(x_1, y_1) < \alpha/2$. We check that the triangle inequality holds. By the symmetry of the distance function we only have to check that

$$\tilde{d}(x,y) \;\le\; \tilde{d}(x,x') \,+\, \tilde{d}(x',y).$$

There are four cases. First if $d(x,y) \ge \alpha/2$ and $d(x',y) \ge \alpha/2$ then

$$\tilde{d}(x,y) \;=\; d(x,y) \;\le\; d(x,x') \,+\, d(x',y) \;=\; \tilde{d}(x,x') \,+\, \tilde{d}(x',y).$$

Second if $d(x,y) \ge \alpha/2$ and $d(x',y) < \alpha/2$ then

$$\tilde{d}(x,y) \;=\; d(x,y) \;\le\; d(x,x') \,+\, d(x',y) \;<\; \tilde{d}(x,x') \,+\, \alpha/2 \;<\; \tilde{d}(x,x') \,+\, \tilde{d}(x',y).$$

Third if $d(x,y) < \alpha/2$ and $d(x',y) \ge \alpha/2$ then

$$\tilde{d}(x,y) \;=\; \alpha/2 \;\le\; d(x',y) \;<\; d(x,x') \,+\, d(x',y) \;=\; \tilde{d}(x,x') \,+\, \tilde{d}(x',y).$$

Finally if $d(x,y) < \alpha/2$ and $d(x',y) < \alpha/2$ then

$$\tilde{d}(x,y) \;=\; \alpha/2 \;=\; \tilde{d}(x',y) \;<\; \tilde{d}(x,x') \,+\, \tilde{d}(x',y).$$

Therefore from isometric embeddings $X \hookrightarrow Z$, $y \hookrightarrow Z$, we have found a metric on $X \coprod Y$, that restricts to the given metrics on X and Y, and so that the Hausdorff distance from X to Y is preserved. This proves that

$$\bar{d}_{GH}(X,\,Y) \;\le\; d_{GH}.$$

□

A map $f : X \to Y$ (not necessarily continuous) between metric spaces is called an ϵ-GHA (for "Gromov-Hausdorff approximation") if $|d_Y(f(x_1), f(x_2)) \,-\, d_X(x_1, x_2)| < \epsilon$ for all $x_1, x_2 \in X$, and Y is in the ϵ-neighborhood of $f(X)$. We can define a new distance function between metric spaces, called $\widehat{d_{GH}}$, by setting

$$\widehat{d_{GH}}(X,Y) \;=\; \inf\{\,\epsilon > 0 \mid \text{there are}\, \epsilon\text{−GHA's}\, f : X \to Y \,\text{and}\, g : Y \to X\,\}.$$

It is a simple exercise to prove that this is a metric: if there is an ϵ_1-GHA $f : X \to Y$ and an ϵ_2-GHA $g : Y \to Z$, then the composition satisfies

$$\begin{aligned} &|d_Z(gf(x_1), gf(x_2)) \,-\, d_X(x_1,x_2)| \\ &\quad\le\; |d_Z(gf(x_1), gf(x_2)) \,-\, d_Y(f(x_1), f(x_2))| \,+\, |d_Y(f(x_1), f(x_2)) \,-\, d_X(x_1,x_2)| \\ &\quad\le\; \epsilon_1 \,+\, \epsilon_2 \end{aligned}$$

and it is also easy to show that the $(\epsilon_1 + \epsilon_2)$-neighborhood of $fg(X)$ is Z. Taking infima, we have that $\widehat{d_{GH}}(X,Z) \;\le\; \widehat{d_{GH}}(X,Y) + \widehat{d_{GH}}(Y,Z)$.

Proposition 3.1.2 *The metrics $\widehat{d_{GH}}$ and d_{GH} are equivalent (though they are not the same).*

Pf

We prove that any sequence that converges in one metric converges in the other. If $X_i \to X$ in the d_{GH} sense, we can easily construct 2ϵ-approximations $f_i : X_i \to X$. To do this, note that for all big enough i, since $X_i \coprod X$ has a metric in which X is in the ϵ-neighborhood of X_i (and vice-versa), we can pick any map that sends a point $p \in X_i$ to some point $f(p) \in X$ a distance at most 2ϵ away from p. This can be done, for instance, by choosing a denumerated, finite set of points $X_F \subset X$ that is ϵ-dense (by the compactness of X), and sending any point $p \in X_i$ to the nearest point of X_F. If two or more points are equally close to p, then send p to the point of X_F that is lower in the denumeration. Then for $p, q \in X_i$ we have $|d(p,q) - d(f(p), f(q))| < 2\epsilon$, and X is clearly in the 2ϵ-neighborhood of $f(X_i)$.

Conversely, if $X_i \to X$ in the $\widehat{d_{GH}}$-topology, we can construct metrics on $X_i \coprod X$, for large enough i in which X is in the 2ϵ-neighborhood of X_i. Construct a distance function d so that if $f_i : X_i \to X$ is a 2ϵ-approximation put $d(x_i, f(x_i)) = \epsilon$, and given any other $x_i \in X_i$ and $x \in X$,

$$d(x_i, x) \;=\; \epsilon + \inf_{x_i' \in X_i} \left(d(x_i, x_i') \,+\, d(f(x_i'), x) \right).$$

We verify the triangle inequality. First assume $x_i, x_i' \in X_i$. The only case to verify is $d(x_i, x_i') \leq d(x_i, x) + d(x, x_i')$, where $x \in X$. We have

$$\begin{aligned}
d(x_i, x) + d(x, x_i') &= 2\epsilon + \inf_{x_i'' \in X_i} \left(d(x_i, x_i'') + d(f(x_i''), x) \right) + \inf_{x_i'' \in X_i} \left(d(x_i', x_i'') + d(f(x_i''), x) \right) \\
&\geq 2\epsilon + \inf_{x_i'' \in X_i} \left(d(x_i, x_i'') + d(f(x_i''), x) + d(x_i', x_i'') + d(f(x_i''), x) \right) \\
&\geq 2\epsilon + d(x_i, x_i') \;\geq\; d(x_i, x_i').
\end{aligned}$$

Next assume $x_i \in X_i$ and $x \in X$. If $x' \in X$ then

$$\begin{aligned}
d(x_i, x) &= \epsilon + \inf_{x_i'' \in X_i} \left(d(x_i, x_i'') + d(f(x_i''), x) \right) \\
&\leq \epsilon + \inf_{x_i'' \in X_i} \left(d(x_i, x_i'') + d(f(x_i''), x') + d(x', x) \right) \\
&= d(x_i, x') + d(x', x),
\end{aligned}$$

and if $x_i' \in X_i$ then

$$\begin{aligned}
d(x_i, x) &= \epsilon + \inf_{x_i'' \in X_i} \left(d(x_i, x_i'') + d(f(x_i''), x) \right) \\
&\leq \epsilon + \inf_{x_i'' \in X_i} \left(d(x_i, x_i') + d(x_i', x_i'') + d(f(x_i'), x) \right) \\
&= d(x_i, x_i') + d(x_i', x).
\end{aligned}$$

Finally assume $x, x' \in X$. Given any $x_i \in X_i$ then

$$\begin{aligned}
d(x, x') &< 2\epsilon + inf_{x_i'' \in X_i} \left(d(f(x_i''), x) + d(f(x_i''), x') \right) \\
&\leq 2\epsilon + inf_{x_i'' \in X_i} \left(d(x_i, x_i'') + d(f(x_i''), x) + d(x_i, x_i'') + d(f(x_i''), x') \right) \\
&\leq 2\epsilon + inf_{x_i'' \in X_i} \left(d(x_i, x_i'') + d(f(x_i''), x) \right) + inf_{x_i'' \in X_i} \left(d(x_i, x_i'') + d(f(x_i''), x') \right) \\
&= d(x, x_i) + d(x_i, x')
\end{aligned}$$

$\square$

3.2 Properties of the Gromov-Hausdorff metric

Proposition 3.2.1 *d_{GH} is a metric on the set of compact metric spaces, modulo isometry.*

Pf

If X and Y are isometric then clearly $d_{GH}(X,Y) = 0$.

Conversely assume $d_{GH}(X,Y) = 0$. Then there is a sequence of distance functions d_i on $X \coprod Y$ with $d_i|_X = d_X$ and $d_i|_Y = d_Y$ so that $d_{i,H}(X,Y) \to 0$. Let $\epsilon_j > 0$ be a sequence that converges to 0. For each j construct finite sets of points $\mathcal{X}_j = \{x_k\}$ and $\mathcal{Y}_j = \{y_k\}$ with the following properties: $\mathcal{X}_j$ is ϵ_j-dense in X, $\mathcal{Y}_j$ is ϵ_j-dense in Y, and for large enough i the sets $\mathcal{X}_j$ and $\mathcal{Y}_j$ are ϵ_j-close in the Hausdorff metric. We also require that $\mathcal{X}_j \subset \mathcal{X}_{j+1}$ and $\mathcal{Y}_j \subset \mathcal{Y}_{j+1}$, so that $\mathcal{X} = \bigcup_j \mathcal{X}_j$ is dense in X and $\mathcal{Y} = \bigcup_j \mathcal{Y}_j$ is dense in Y.

Now consider the distance functions $\{d_i\}$ restricted to $\mathcal{X}_j \cup \mathcal{X}_j$. Because this set is finite, a subsequence d_{i_j} converges to a limiting pseudometric $\overline{d}_j$. Passing to ever more refined subsequences of d_i as j increases and taking a diagonal subsequence (which we also call d_i), we get convergence to a pseudometric d on $\mathcal{X} \cup \mathcal{Y}$, a dense subset of $X \coprod Y$, and therefore convergence on $X \coprod Y$.

Given any ϵ_j, a given point $x \in X$ is ϵ_j-close to a point $x_j \in \mathcal{X}$, which is ϵ_j-close to a point of $y_j \in \mathfrak{Y}$. Taking a limit $y = \lim_j y_j$ we have that $d(x,y) = 0$. Similarly given an arbitrary point $y \in Y$ we can find a point $x \in X$ with $d(x,y) = 0$.

Identify points $a, b \in X \coprod Y$ with $d(a,b) = 0$ and call the moduli space M. Since $d|_X = d_X$ and $d|_Y = d_Y$ and the triangle inequality holds, a point in X is identified with a unique point in Y, and vice-versa. We now have natural isometric equivalences $X \to M$ and $Y \to M$ so that X and Y are isometric. $\square$

Proposition 3.2.2 *The Gromov-Hausdorff topology on the set of compact metric spaces is second countable.*

Pf

If a topology is Hausdorff and separable it is second countable, or better, a separable metric space is second countable. Consider the set $\tilde{\mathcal{X}}$ of finite metric spaces where all distances are rational. There are countably many such spaces. To see that these spaces are dense, consider a compact metric space X. We can construct a sequence X_i of such finite spaces that converge to X by letting $X_i \subset X$ be a 2^{-i}-dense set of points. The metric space X_i has a distance of less than 2^{-i} from some finite metric space $\tilde{X}_i$ with rational distance, so that $\tilde{X}_i \to X$. This proves that $\tilde{\mathcal{X}}$ is dense. $\square$

As it happens, the pointed Gromov-Hausdorff distance is not second countable. We have already constructed an uncountable collection of subsets of $\mathbb{R}^1$ that are Hausdorff distance 1 from each other.

Lemma 3.2.3 (Gromov's Precompactness Lemma) *Let* $N : \mathbb{N} \to \mathbb{N}$ *be monotonic. Assume* $\mathfrak{M}$ *is a collection of metric spaces so that each* $M \in \mathfrak{M}$ *has a* $\frac{1}{j}$*-dense discrete subset of cardinality* $\leq N(j)$*. Then* $\mathfrak{M}$ *is precompact.*

Pf
Let $\{M_i\} \subset \mathfrak{M}$, and let $\tilde{M}_{i,j} \subset M_i$ be a $\frac{1}{j}$-dense subset of cardinality $\leq N(j)$. By replacing $N(j)$ with $\sum_{i=1}^{j} N(i)$ we can assume that $\tilde{M}_{i,j} \subset \tilde{M}_{i,j+l}$. Fixing j and letting $i \to \infty$ we get convergence of $\tilde{M}_{i,j}$ along a subsequence to a space $\tilde{M}_j$. Passing to further refinements of the subsequence and taking a diagonal sequence, we get a sequence of distance functions d_k that converge on each $\tilde{M}_j$, and therefore on $\tilde{M} = \overline{\bigcup_j \tilde{M}_j}$. Now given $\epsilon > 0$ there is an i so that $\tilde{M}_i$ is ϵ-close to $\tilde{M}$, and there is a j so that $M_{i,j}$ is ϵ-close to both $\tilde{M}_i$ and to M_i. Thus M_i converges to $\tilde{M}$. $\square$

Chapter 4

Convergence Theorems

February 9, 2010

4.1 Volume comparison and the Heintze-Karcher theorem

Given a complete embedded submanifold $N^k \subset M^n$, we can parametrize M locally near N by some neighborhood of N in $N^k \times \mathbb{R}^{n-k}$, via the normal exponential map. This requires identifying $N^k \times \mathbb{R}^{n-k}$ with the normal tangent bundle $T^\perp N$, which can be done locally. Namely fixing a point $p \in N$, one identifies $T_q^\perp N$ with $T_p^\perp N$ by parallel transport in the normal bundle, whenever there is a unique geodesic from p to q. Putting coordinates $\{x^1, \ldots, x^k\}$ on N near p and coordinates $\{x^{k+1}, \ldots, x^n\}$ on $\mathbb{R}^{n-k}$, we can express the metric tensor g_{ij} and the volume form $dVol = \sqrt{\det g_{ij}} dx^1 \wedge \cdots \wedge dx^n$ in components.

Lemma 4.1.1 (Heintze-Karcher (1978)) *Assume $N^k \subset M^n$ is an embedded submanifold. Let ξ be a geodesic perpendicular to N. If all sectional curvatures along ξ are $\geq \kappa$ and the mean curvature vector of N^k at $\xi(0)$ is H, then*

$$\sqrt{\det g_{ij}(t)} \;\leq\; f_{n,k,\kappa,H}(t),$$

where t measures the Riemannian distance to N, and $f_{n,k,\kappa,H}$ is a function uniquely determined by n, k, κ, and H. This inequality holds until when $t \geq 0$ and until $\det \mathcal{A}(t, \xi)$ *has a zero.*

Pf

Let $\{x^1, \ldots, x^n\}$ be coordinates near $p \in N$, as above. The identification of nearby normal tangent spaces implies

$$\nabla_{\partial/\partial x^i}\frac{\partial}{\partial x^j} \;=\; -B_{\partial/\partial x^j}\left(\frac{\partial}{\partial x^i}\right)$$

where $1 \leq i \leq k$ and $k+1 \leq j \leq n$. Now assume $x^{k+1}, \ldots, x^n$ are spherical coordinates with $x^n = t$ being the radial coordinate.

Letting $\xi(t)$ be a geodesic with initial direction perpendicular to N at p, we can choose frames $\{E_1, \ldots, E_n\}$ along ξ so that $E_i = \frac{\partial}{\partial x^i}$ for $1 \leq i \leq k$ and so that $\nabla_{\partial/\partial t}\frac{\partial}{\partial x^j} = E_j$ for $k+1 \leq j \leq n-1$. Let $A_{ij} = g\left(\frac{\partial}{\partial x^i}, E_j\right)$, $1 \leq i, j \leq n-1$. It is easy to prove that $\det A_{ij} \;=\; \sqrt{\det g_{ij}}$.

Letting $\mathcal{R}(t)$ be the linear operator

$$(\mathcal{R}A)_{ij} \;=\; g\left(\mathrm{Rm}\left(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial t}\right)\frac{\partial}{\partial t},\, E_j\right),$$

we have that $\frac{d^2}{dt^2}A_{ij} \;+\; (\mathcal{R}A)_{ij} \;\;=\;\; 0$. This system of differential equations has initial conditions

$$\begin{aligned} A(0) \;&=\; \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix} \\ A'(0) \;&=\; \begin{pmatrix} -B_{\partial/\partial t} & 0 \\ 0 & I \end{pmatrix} \end{aligned}$$

Now consider the solution $B(t)$ of $B'' + \kappa B \;=\; 0$ with initial conditions $B(0) = A(0)$, $B'(0) = A'(0)$. The Index theorem implies (for instance) that the trace of $A(t)$ is less than or equal to the trace of $B(t)$. Then $\det(A)^{1/n-1} \;\leq\; \frac{1}{n-1}Tr(A) \;\leq\; \frac{1}{n-1}Tr(B) \;=\; \det(B)^{1/(n-1)}$. □

Corollary 4.1.2 (Cheeger's lemma (1970)) *Assume M is a compact Riemannian manifold. Then* inj_M *is bounded from below by a constant determined by* $\mathrm{Vol}(M)$, $\mathrm{Diam}(M)$, *and* $\kappa = \min_{p\in M}\sec(M)$.

Pf

Given a small geodesic lasso, there is a geodesic loop of shorter (or equal) length in its free homotopy class. Such a loop ξ has mean curvature zero, so integrating the Heinze-Karcher inequality gives

$$\mathrm{Vol}(M) \leq C(n, \kappa, \mathrm{Diam}(M))\, l(\xi).$$

□

Theorem 4.1.3 (Bishop-Gromov volume comparison) *If* $\mathrm{Ric} \geq (n-1)H$ *for some real number* H, *then* $\mathrm{Vol}\, B_p(r) \;\leq\; \mathrm{Vol}\, B^H(r)$, *where* $B^H(r)$ *indicates the ball of radius* r *in the spaceform of constant sectional curvature* H.

Pf

The Weitzenböck formula

$$\frac{1}{2}\triangle|\nabla f|^2 \;=\; |\nabla^2 f|^2 \;+\; \langle \nabla f, \nabla \triangle f\rangle \;+\; \mathrm{Ric}(\nabla f, \nabla f) \tag{4.1}$$

gives, after plugging in a distance function r,

$$0 \;=\; |\nabla^2 r|^2 \;+\; \frac{\partial}{\partial r}\triangle r \;+\; \mathrm{Ric}(\nabla r, \nabla r). \tag{4.2}$$

Now use spherical coordinates $\{r, x^2, \ldots, x^n\} = \{r, x\}$ and set

$$A = \sqrt{\det(g_{ij})}.$$

Then

$$\frac{\partial^2}{\partial r^2}\log A \;=\; \frac{\partial}{\partial r}\triangle r \;=\; -|\nabla^2 r|^2 \;-\; \mathrm{Ric}(\nabla r, \nabla r) \tag{4.3}$$

So with $J^{n-1} = A$, we get

$$\frac{\partial}{\partial r^2}\log J \;=\; -\frac{1}{n-1}|\nabla^2 r|^2 \;-\; \frac{1}{n-1}\mathrm{Ric}(\nabla r, \nabla r) \tag{4.4}$$

$$\frac{J''}{J} - \left(\frac{J'}{J}\right)^2 \;=\; -\frac{1}{n-1}|\nabla^2 r|^2 \;-\; \frac{1}{n-1}\mathrm{Ric}(\nabla r, \nabla r) \tag{4.5}$$

and with

$$\frac{\partial}{\partial r}\log J \;=\; \frac{1}{n-1}\frac{\partial}{\partial r}\log A \;=\; \frac{\triangle r}{n-1}, \tag{4.6}$$

we get another "Heintze-Karcher inequality"

$$(n-1)\,\frac{J''}{J} \;=\; \frac{(\triangle r)^2}{n-1} \;-\; |\nabla^2 r|^2 \;-\; \mathrm{Ric}(\nabla r, \nabla r) \;\leq\; -\mathrm{Ric}(\nabla r, \nabla r) \;\leq\; -(n-1)H \tag{4.7}$$

Conversely, on the space form of sectional curvature H we get exactly the equation $J'' + HJ = 0$. Comparing these differential equations and integrating gives the desired inequality. □

4.2 Convergence Theorems for Noncollapsed manifolds

In 1967 Cheeger proved the following finiteness theorem

Theorem 4.2.1 (Cheeger's Diffeofiniteness) *Let $\mathcal{M}(n, \Lambda, \delta, \nu)$ be the set of Riemannian manifolds of dimension n, sectional curvature $|\sec| < \Lambda$, diameter $\mathrm{diam} < \delta$, and volume $\mathrm{Vol} > \nu$. Then there are only finitely many diffeomorphism types of manifolds in $\mathcal{M}(n, \Lambda, \delta, \nu)$.*

Idea of Pf First a lower bound on the injectivity radius is established. Given a manifold $M \in \mathcal{M}$ one establishes that finitely many Euclidean charts of definite size can cover the manifold, where each chart is given "Riemann normal coordinates". Finally one shows that the transitions between different "normal coordinate" regimes is controlled by the curvature. With control over the number of charts and the transition functions between them, the finiteness of diffeomorphism classes follows. $\square$

Theorem 4.2.2 (Gromov's Precompactness Theorem) *Let $\mathcal{M}(n, \lambda, \nu, D)$ be the set of compact manifolds of dimension n of volume greater than ν, diameter less than D, and Ricci curvature greater than λ. Then $\mathcal{M}(n, \nu, \lambda)$ is precompact in the pointed Gromov-Hausdorff topology.*

Pf
Let $\{(M_i, p_i)\} \subset \mathcal{M}(n, \lambda)$ be a sequence of such manifolds. Choose a maximal $\frac{1}{j}$-separated set of points in M (in particular, this set is $\frac{1}{j}$-dense). Consider the (disjoint) balls of radius $\frac{1}{2j}$ around each point. Each has volume

$$\begin{aligned}
\operatorname{Vol} B(p, 1/2j) &\geq \frac{\operatorname{Vol} B^{\Lambda}(1/2j)}{\operatorname{Vol} B^{\Lambda}(2D)} \operatorname{Vol} B(p, 2D) \\
&\geq \frac{\operatorname{Vol} B^{\Lambda}(1/2j)}{\operatorname{Vol} B^{\Lambda}(2D)} \operatorname{Vol} B(p_i, D) \\
&= C(j, R, \Lambda) \operatorname{Vol} B(p_i, D).
\end{aligned}$$

Thus there are fewer than $N(j, D, \Lambda) = 1/C(j, D, \Lambda)$ points in our maximal $\frac{1}{j}$-separated set. Now Gromov's Precompactness Lemma states that the balls $B(p_i, D) \subset M_i$ converge along a subsequence to some metric space set $\tilde{M}$. $\square$

In 1981 Gromov extended this result to prove precompactness in the Lipschitz topology. Specifically

Theorem 4.2.3 (Gromov's $C^{1,1}$-precompactness) *The space $\mathcal{M}(n, \Lambda, \delta, \nu)$ is precompact in the Lipschitz topology. Any sequence of manifolds $\{M_i\} \in \mathcal{M}(n, \Lambda, \delta, \nu)$ converges along a subsequence to a differentiable manifold with a C^0 metric, and a $C^{1,1}$ distance function.*

In 1987 this theorem was improved (independently) by S. Peters and Greene-Wu. We have

Theorem 4.2.4 ($C^{1,\alpha}$-precompactness) *The space $\mathcal{M}(n, \Lambda, \delta, \nu)$ is precompact in the Lipschitz topology. Any sequence of manifolds $\{M_i\} \in \mathcal{M}(n, \Lambda, \delta, \nu)$ converges along in the $C^{1,\alpha}$ topology to a differentiable manifold with a $C^{1,\alpha}$ metric.*

This theorem holds locally in appropriate settings; for instance the proof goes through almost unchanged on star-shaped domains. Better control over the curvature tensor yields improved convergence: if the Riemann tensors are bounded in the C^k sense, then convergence is in the $C^{k+1,\alpha}$ topology. In a sense this theorem is a kind of global (though slightly weakened) Arzela-Ascoli or Rellich theorem.

We have already discussed spaces with Ricci curvature bounded from below in the context of Gromov's precompactness theorem. In some cases Gromov's theorem can be sharpened.

Theorem 4.2.5 (Anderson-Cheeger) *The space of compact n-dimensional manifolds with (possibly negative) lower bounds on Ricci curvature, (positive) lower bounds on injectivity radius and upper bounds on volume is precompact in the Gromov-Hausdorff topology. Sequences of such manifolds subconverge in the Gromov-Hausdorff distance to manifolds in the $C^{1,\alpha}$-differentiability class, and metrically converge in the $C^{0,\alpha}$-topology.*

Theorem 4.2.6 (Cheeger-Colding) *The space of compact n-dimensional manifolds with definite lower bounds on Ricci curvature, (positive) lower bounds on the volume of unit balls and upper bounds on volume is precompact in the Gromov-Hausdorff topology. Off a singular set of codimension 2 or greater, convergence is in the $C^{0,\alpha}$-topology to a connected differentiable manifold.*

Chapter 5

The Bieberbach Theorem I

February 16, 2010

5.1 Compact flat manifolds: Bieberbach's theorem on crystallographic groups

A discrete subgroup G of the group of Euclidean motions $\subset \mathcal{O}(n) \ltimes \mathbb{R}^n$ is called a crystallographic group if it acts freely and has compact fundamental domain. A crystallographic group determines a compact flat manifold, and a compact flat manifold is determined by (at least one) crystallographic group. Certain fundamental questions arise. Given n, are there finitely many flat manifolds of dimension n? Is any flat manifold covered by a torus? Are these coverings normal? This can be rephrased: given a crystallographic group G, does it have a normal Abelian subgroup of finite index and maximal rank? Bieberbach answered this in the affirmative.

Given a motion $\alpha \in \mathcal{O}(n) \ltimes \mathbb{R}^n$ we can project to its rotational part $A = A(\alpha)$ and its translational part $t = t(\alpha)$. If A is a transformation we can write $\mathbb{R}^n = E_0 \oplus \cdots \oplus E_k$ where A acts as a rotation of angle θ_i on E_i, and possibly acts as a reflection on E_k (this is to account for the nonorientable case). The θ_i are called the *principle rotational angles*.

Theorem 5.1.1 (Bieberbach, 1911) *Assume $G \subset \mathcal{O}(n) \ltimes \mathbb{R}^n$ acts freely on $\mathbb{R}^n$. If $\alpha \in G$ then all principle rotational angles of $A(\alpha)$ are rational. If the translational parts of G span some subspace $S \subset \mathbb{R}^n$, then the pure translations of G span E.*

Since the translational parts of elements of a crystallographic group spans $\mathbb{R}^n$, it follows that a flat manifold is covered by a torus. Bieberbach was also able to use this with a theorem on group extensions to prove that there are only finitely many quotients.

As a special case of his almost-flat manifold theorem, Gromov proved a stronger form of Bieberbach's theorem.

Theorem 5.1.2 *Let G be a crystallographic group. Then*

i) There is a translational, normal subgroup $\Gamma \lhd G$ of finite index (indeed $ind(\Gamma : G) < 2(4\pi)^{\frac{1}{2}n(n-1)}$).

ii) If $\alpha \in G$ then either α is a translation or the smallest nonzero principle angle $A(\alpha)$ is greater than $\frac{1}{2}$.

iii) Further, if $\alpha \in G$ and $0 < \theta_1 < \cdots < \theta_k$ are the nonzero principle rotational angles of $A = A(\alpha)$, then

$$\theta_l \geq \frac{1}{2}(4\pi)^{l-k}$$

Via (i), this formulation directly shows that there are finitely many flat manifolds of a given dimension.

5.2 Statement of Gromov's theorem on almost flat manifolds

Gromov's far-reaching extension of Bieberbach's theorem states that almost-flat manifolds have a finite normal cover isomorphic to a nilmanifold. Specifically he proved

Theorem 5.2.1 (Gromov 1978) *Let M^n be a compact Riemannian manifold and set $K = max|\sec(M)|$ and $d = \mathrm{diam}(M)$. If $d^2K < exp(-exp(n^2))$, then M^n is covered by a nilmanifold. More specifically,*

- *$\pi_1(M)$ contains a torsion-free nilpotent normal subgroup Γ or rank n,*
- *The quotient $G = \Lambda\backslash\pi_1(M)$ has order $\leq 2(6\pi)^{\frac{1}{2}n(n-1)}$ and is isomorphic to a subgroup of $\mathcal{O}(n)$,*
- *The finite covering of M with fundamental group Γ and deckgroup G is diffeomorphic to a nilmanifold $\Gamma\backslash N^n$, and*
- *The simply connected Lie group N is uniquely determined by $\pi_1(M)$.*

Gromov claims his proof is a generalization of Bieberbach's proof, although such a view is hard to support considering his introduction of several radical new techniques. Ruh (1982) proved that collapsed manifolds are actually *infranil.* A compact manifold M is called infranil if it is a quotient of a nilpotent Lie group N by affine transformations such that the image of the holonomy action of the canonical flat affine connection is finite.

5.3 Finsler geometry on $\mathcal{O}(n)$

Proposition 5.3.1 *Given an operator $A \in \mathcal{O}(n)$, there is a decomposition*

$$\mathbb{R}^n \;=\; E_0 \oplus E_1 \oplus \ldots \oplus E_k$$

where A acts as a simple rotation through $\pm\theta_i$ on E_i, and we arrange

$$0 \;=\; \theta_0 \;<\; \theta_1 \;<\; \ldots \;<\; \theta_k$$

(possibly $E_0 = \{0\}$). If A is orientation-reversing then E_k is 1-dimensional and we put $\theta_k = \pi$.

Pf The E_i are the eigenspaces of A. The $\pm\theta_i$ are the corresponding eigenvalues. □

We call the θ_i the *principle angles* of $A \in SO(n)$. Define $|A|$ by

$$|A| \;=\; \theta_k \;=\; \max_{x \in \mathbb{R}_n} \measuredangle(x, Ax).$$

A norm on $\mathfrak{so}(n)$ (the operator norm) can be defined by stating that for $X \in \mathfrak{so}(n)$, we have

$$|X| \quad = \quad \max\{\, |Xv| \mid v \in \mathbb{R}^n \text{ and } |v| = 1 \,\}.$$

Of course this is the magnitude of the largest eigenvalue of a. Note that this norm is Ad-invariant:

$$\begin{aligned} |Ad_A X| &= \max\{\, |AXA^{-1}v| \mid v \in \mathbb{R}^n \text{ and } |v| = 1 \,\} \\ &= \max\{\, |XA^{-1}v| \mid v \in \mathbb{R}^n \text{ and } |v| = 1 \,\} \\ &= \max\{\, |Xw| \mid w \in \mathbb{R}^n \text{ and } |Aw| = 1 \,\} \\ &= \max\{\, |Xw| \mid w \in \mathbb{R}^n \text{ and } |w| = 1 \,\} \;=\; |a|. \end{aligned}$$

Proposition 5.3.2 *Left-translating the $|\cdot|$ norm on $\mathfrak{so}(n)$ to each tangent space on $SO(n)$ gives a Finsler metric, with (right) invariant distance function*

$$d(A, B) \;=\; |AB^{-1}|.$$

If $X \in \mathfrak{so}(n)$ has $|X| \leq \operatorname{diam}(SO(n)$ and $A = \exp(X)$, then

$$|X| \;=\; |A|.$$

Pf

The Finsler metric is obtained by left-translation, so therefore the distance function will be *right* invariant. Writing $d(A,B) = d(AB^{-1}, Id)$, we only have to verify that $d(C, Id) = |C|$. Let C be any element of $SO(n)$, with principle rotational angles $\theta_k, \theta_{k-1}, \ldots$. Let $C(t)$, $t \in [0,1]$, be the path in $SO(n)$ consisting of matrices with the eigenspace decomposition of C, but with the primary rotational angles given by $t\theta_k$. Then $|\dot{C}(t)| = \theta_k$ and so clearly $L(C(t)) = \int_0^1 |\dot{C}(t)|dt = \theta_k = |C|$. Therefore $\mathrm{dist}(C, Id) \le |C|$. We can show that $C(t)$ is a distance-minimizing path. Let $\mathcal{C}(t)$ be another path with $\mathcal{C}(0) = Id$, $\mathcal{C}(1) = C$, and principle rotation angles $\theta_k(t)$. Given any such path we can create a new path $\tilde{\mathcal{C}}(t)$ with the same principle rotational angles, but with the same eigenspace decomposition of C. Since $|\dot{\tilde{\mathcal{C}}}(t)| = |\dot{\mathcal{C}}(t)| = |\dot{\theta}_k(t)|$, these paths have the same pathlengths. With $L(\mathcal{C}(t)) = \int_0^1 |\dot{\theta}_k(t)|dt$, we see that this is minimized when θ_k is linear.

To prove the last statement, note that $\gamma(t) = \exp(tX)$ realizes the minimum distance from Id to A. Since $\dot{\gamma}(t) = dL_{\gamma(t)}X$ we have $|\dot{\gamma}(t)| = |X|$ so that $|A| = \int_0^1 |\dot{\gamma}(t)|dt = |X|$. $\square$

In this Finsler metric there are is no point with a unique minimizing path joining it to the origin. To see this note that since pathlength depends only on the largest principle rotational angle, the other rotational angles can be modified in any way, and as long as they remain less than the largest angle, pathlength will be unaffected.

Given $A, B \in SO(n)$ let $K_A : SO(n) \to SO(n)$ act on B by conjugation: $K_A(B) = ABA^{-1}$. Since $K_A(Id) = Id$ we can regard $dK_A : \mathfrak{so}(n) \to \mathfrak{so}(n)$. We can also define $\mathrm{Ad}_A : \mathfrak{so}(n) \to \mathfrak{so}(n)$ by $\mathrm{Ad}_A X = AXA^{-1}$.

Lemma 5.3.3 *Given $A \in SO(n)$ and $X \in \mathfrak{so}(n)$, we have*

$$dK_A(X) = \mathrm{Ad}_A(X) \tag{5.1}$$

$$\exp(\mathrm{Ad}_A X) = K_A(\exp(X)) \tag{5.2}$$

$$\mathrm{Ad}_{\exp(tY)}X = \mathrm{Exp}(t\,\mathrm{ad}_Y)X \triangleq \sum_{i=0}^{\infty} \frac{1}{i!}\,(t\,\mathrm{ad}_Y)^i\,X. \tag{5.3}$$

Pf

Equation (5.1) follows from taking the derivative

$$\left.\frac{d}{dt}\right|_{t=0} \mathrm{Ad}_A \exp(tX) = AXA^{-1}.$$

Keeping in mind that $SO(n)$ is a matrix group, expression (5.2) is just $e^{AXA^{-1}} = Ae^X A^{-1}$. One way to prove (5.3) is to find a Taylor series expression for $\mathrm{Ad}_{\exp(tY)}$, and to prove that the radius of convergence is infinite. We have

$$\begin{aligned}
\frac{d^k}{dt^k}\Big|_{t=0} \mathrm{Ad}_{\exp(tY)} X &= \frac{d^{k-1}}{dt^{k-1}}\Big|_{t=0} (Y \exp(tY) X \exp(-tY) - \exp(tY) X \exp(-tY) Y) \\
&= \frac{d^{k-1}}{dt^{k-1}}\Big|_{t=0} \mathrm{Ad}_{\exp(tY)} \mathrm{ad}_Y X \\
&= (\mathrm{ad}_Y)^k X.
\end{aligned}$$

The radius of convergence is infinite because the matrix $(\mathrm{ad}_Y)^k X$ is a polynomial expression of order k in the entries of Y and of order 1 in the entries of X. □

Proposition 5.3.4 *The exponential map* $\exp : \mathfrak{so}(n) \to SO(n)$ *is length-nonincreasing, as measured in the Finsler norm.*

Pf

In addition to the Finsler norm is the bi-invariant metric g on $SO(n)$, the geodesics of which are precisely the left- or right-translates of paths of the form $\exp(tY)$, $Y \in \mathfrak{so}(n)$. If ∇^L is the canonical left-invariant connection (zero on any left-invariant vector field), the Riemannian connection is

$$\nabla_X Y \;=\; \frac{1}{2}\left(\nabla^L_X Y \,+\, \nabla^L_Y X\right) \,+\, \frac{1}{2}[X, Y].$$

Let $\gamma(t)$ be a geodesic with initial direction X, meaning $\gamma(t) = \exp(tX)$. Let $J_Y(t)$ be the Jacobi field along γ with initial conditions $J_Y(0) = 0$ and $\nabla_{\dot\gamma} J_Y(0) = Y$. The map $d\exp_X : \mathfrak{so}(n) \to T_{\exp(X)} SO(n) \approx \mathfrak{so}(n)$ is just $J_Y(1)$. Also, $d\exp_{tX} Y \;=\; \frac{1}{t} J_Y(t)$.

Now consider the Jacobi equation $0 = \nabla_{\dot\gamma}\nabla_{\dot\gamma} Y + R(Y, \dot\gamma)\dot\gamma = \nabla_{\dot\gamma}\nabla_{\dot\gamma} Y - \frac{1}{4}[\dot\gamma, [\dot\gamma, Y]]$. Since $\dot\gamma$ is left-invariant, for any vector field Y along γ one easily checks

$$\nabla_{\dot\gamma}\nabla_{\dot\gamma} Y \,+\, R(Y, \dot\gamma)\dot\gamma \;=\; \frac{1}{4}\nabla^L_{\dot\gamma}\nabla^L_{\dot\gamma} Y \,+\, \frac{1}{2}\left[\dot\gamma, \nabla^L_{\dot\gamma} Y\right],$$

so the Jacobi equation can be written $\nabla^L_{\dot\gamma}\nabla^L_{\dot\gamma} Y + 2\left[\dot\gamma, \nabla^L_{\dot\gamma} Y\right] = 0$. Putting $K(t) = dL_{\gamma(t)^{-1}} J(t) \in \mathfrak{so}(n)$, the Jacobi equation reads

$$\ddot K \,+\, 2[X, \dot K] \;=\; 0.$$

Thus $\dot K(t) \;=\; \mathrm{Exp}(2t\,\mathrm{ad}_X) \cdot \dot K(0) \;=\; \mathrm{Ad}_{\exp(2tX)} Y$, and since $|\dot K(t)| \;=\; |\dot K(0)| \;=\; |Y|$ and $|\dot J_Y(t)| = |\dot K(t)|$, we have

$$|J_Y(t)| = |K(t)| \;\leq\; \int_0^t |\dot K(t)| dt \;=\; t|\dot K(0)| \;=\; t|Y|.$$

Therefore $|d\exp_{tX} Y| \;=\; \frac{1}{t}|J_Y(t)| \;\leq\; |Y|$, and so the exponential map is length-nonincreasing.
□

Proposition 5.3.5 *Given $A, B \in \mathcal{SO}(n)$, we have $d([A,B],\, Id) \le 2d(A,\, Id)d(B,\, Id)$.*

Pf

Putting $A = \exp(X)$ and $B = \exp(Y)$, we can connect A with BAB^{-1} with the curve

$$\gamma(t) \;=\; \exp\left(\mathrm{Exp}(t\,\mathrm{ad}_Y)X\right) \qquad t \in [0,1].$$

Now

$$d([A,B],\, Id) \;=\; d(A,\, BAB^{-1}) \;\le\; \int_0^1 |\dot\gamma(t)|\, dt.$$

Estimating $|\dot\gamma(t)|$ we have

$$\begin{aligned} |\dot\gamma(t)| \;&=\; \left|\frac{d}{dt}\exp\left(\mathrm{Exp}(t\,\mathrm{ad}_Y)X\right)\right| \;=\; \left|d\exp_{\mathrm{Exp}(t\,\mathrm{ad}_Y)X}\left(\frac{d}{dt}\mathrm{Exp}(t\,\mathrm{ad}_Y)X\right)\right| \\ &\le\; \left|\frac{d}{dt}\left(\mathrm{Exp}(t\,\mathrm{ad}_Y)X\right)\right| \;=\; \left|\,\mathrm{Exp}\left(t\,\mathrm{ad}_Y\right)\,[Y,X]\,\right| \\ &=\; \big|[Y,X]\big| \;\le\; 2|X||Y| \;=\; 2|A||B|. \end{aligned}$$

□

Corollary 5.3.6 *A discrete subgroup of $SO(n)$ generated by elements of norm less than $1/2$ is nilpotent.*

Pf

On generators, $|[A,B]| < \min\{|A|,|B|\}$. Chains of commutators therefore converge on the identity, and since the subgroup is assumed discrete, any such chain must eventually terminate with the identity element. □

Now define a norm on the group of Euclidean symmetries $\mathcal{O}(n) \ltimes \mathbb{R}^n$ by

$$|\alpha| \;\triangleq\; \max\{\,|r(\alpha)|,\; c\cdot|t(\alpha)|\,\}. \tag{5.4}$$

Proposition 5.3.7 *Any set of Euclidean motions $\{\alpha_1,\ldots,\alpha_k\}$ with $d(\alpha_i,\alpha_j) \ge \max\{|\alpha_i|,|\alpha_j|\}$ has $k \le 3^{n+\frac{1}{2}n(n-1)}$.*

Pf We will work in the tangent space $\mathfrak{so}(n) \times \mathbb{R}^n$ at the identity. Let $A_i = r(\alpha_i)$ and $a_i = t(\alpha_i)$, and let $S_i \in \mathfrak{so}(n)$ be such that $\exp S_i = A_i$. Define a norm on $\mathfrak{so}(n) \times \mathbb{R}^n$ by $|(S,a)| = \max\{|S|, c|a|\}$, where c is any constant. Put $w_i = (S_i, a_i)$. Then $\tilde w_i = \frac{w_i}{|w_i|}$ lie on the unit sphere, and

$$\begin{aligned} |\tilde w_i \,-\, \tilde w_j| \;&\ge\; \left|\frac{w_i}{|w_j|} \,-\, \tilde w_j\right| \,-\, \left|\frac{w_i}{|w_j|} \,-\, \tilde w_i\right| \\ &=\; \frac{1}{|w_j|}\left(|w_i \,-\, w_j| \,-\, \big||w_i| \,-\, |w_j|\big|\right). \end{aligned}$$

By using $|d \exp_X Y| \leq |Y|$ from above, it is a simple matter to show $d(\alpha_i, \alpha_j) \leq |w_i - w_j|$. Also, $|w_i| = |\alpha_i|$, so therefore

$$|\tilde{w}_i - \tilde{w}_j| \quad \geq \quad \frac{1}{|\alpha_j|} \left(\max\{|\alpha_i|, |\alpha_j|\} - \big||\alpha_i| - |\alpha_j|\big|\right) = 1.$$

Thus we have found points on the unit sphere in $\mathfrak{so}(n) \times \mathbb{R}^n$ with unit mutual separation. There is a uniform upper bound on how many such points there can be in any such collection.
□

Chapter 6

The Bieberbach Theorem II

February 18, 2010

Let G be a crystallographic group. Given $A \in G \subset \mathcal{O}(n) \ltimes \mathbb{R}^n$, recall the norm

$$|A| \;=\; \max\{|r(\alpha)|,\, c|t(\alpha)|\} \tag{6.1}$$

where the norm $|\cdot|$ on $\mathcal{O}(n)$ was defined previously, and the norm $|\cdot|$ on $\mathbb{R}^n$ is the distance to the origin.

6.1 Two lemmas

Put $G_\rho = \{\alpha \in G \,|\, |t(\alpha)| < \rho\}$, and $G_\rho^\epsilon = \{\alpha \in G \,|\, |t(\alpha)| < \rho_i,\, |r(\alpha)| < \epsilon\}$. We will use the notation $B(s)$ to indicate the ball of radius s centered at the origin in $\mathbb{R}^n$. We will use "d" to indicate the Finsler metric on $\mathcal{O}(n)$ and "dist" to indicate the Euclidean distance function on $\mathbb{R}^n$.

The most important consequence of the following lemma is that, regardless of how small ϵ is, the pseudogroup G_ρ^ϵ has plenty of elements if ρ is big enough.

Lemma 6.1.1 *Given $R > 0$ and $\epsilon \in \left(0, \frac{1}{2}\right)$, there is some $\rho > R$ such that the set translation parts $t(\alpha)$ of elements $\alpha \in G_\rho^\epsilon$ is $\rho/4$-dense in $B(3\rho/4)$.*

<u>*Pf*</u>

Let r be the radius of the fundamental domain, and put $\rho_i = (R + r) \cdot 10^i$. If the lemma is false, there is an $x_i \in B(\rho_i)$ such that $\mathrm{dist}(x_i, t(G_{\rho_i}^\epsilon)) > \rho_i/4$. However the set of translational parts in G_{ρ_i} is r-dense in $B(3\rho_i/4)$, so there is some $\alpha_i \in G_{\rho_i}$ so that $|t(\alpha_i) - x_i| \le r$.

If the lemma is false we prove that the rotational parts $r(\alpha_1), r(\alpha_2), \ldots$ are all ϵ-separated from each other. Using $A_i = r(\alpha_i)$ and $a_i = t(\alpha_i)$, we compute $t(\alpha_i\alpha_j^{-1}) = -A_iA_j^{-1}a_j + a_i$. Then, with $i > j$,

$$|t(\alpha_i\alpha_j^{-1})| \;\le\; |A_iA_j^{-1}a_j| + |a_i - x_i| + |x_i| \;\le\; \rho_j + r + \frac{3}{4}\rho_i \;<\; \rho_i,$$

so that $\alpha_i\alpha_j^{-1} \in G_{\rho_i}$. Also

$$|t(\alpha_i\alpha^{-1}) - x_i| \;\le\; |A_iA_j^{-1}a_j| + |a_i - x_i| \;<\; \rho_{i-1} + r \;<\; \frac{\rho_i}{4},$$

so that $\alpha_i\alpha_j^{-1} \notin G^{\epsilon}_{\rho_i}$. Therefore $|r(\alpha_i\alpha_j^{-1})| \ge \epsilon$. Thus we have a sequence $r(\alpha_1), r(\alpha_2), \ldots$ of ϵ-separated elements of $\mathcal{O}(n)$, an impossibility. □

Lemma 6.1.2 *If $\epsilon < \frac{1}{2}$, then the group $\langle G^{\epsilon}_{\rho}\rangle$ generated by G^{ϵ}_{ρ} is d-nilpotent with $d = d(n)$.*

Pf

Fix the constant in (6.1) to be ϵ/ρ, so that $|\alpha| < \epsilon$ iff $\alpha \in G^{\epsilon}_{\rho}$. Define a *short basis* $\{\alpha_1, \ldots, \alpha_d\}$ inductively by selecting $\alpha_1 \in G^{\epsilon}_{\rho}$ so that α_1 has minimal norm, and selecting an element $\alpha_i \in G^{\epsilon}_{\rho} - \langle\alpha_1, \ldots, \alpha_{i-1}\rangle$ of minimal norm among all elements of $G^{\epsilon}_{\rho} - \langle\alpha_1, \ldots, \alpha_{i-1}\rangle$.

We first prove that $d(\alpha_i, \alpha_j) \ge \max\{|\alpha_i|, |\alpha_j|\}$; a lemma of Finsler geometry now directly gives that $d \le 3^{n^2}$. Wlog put $i > j$. Arguing for a contradiction, assume $|\alpha_i\alpha_j^{-1}| = d(\alpha_i, \alpha_j) < |\alpha_i| \le \epsilon$. It follows from the definitions that $\alpha_i\alpha_j^{-1} \in G^{\epsilon}_{\rho}$, but since α_i was chosen minimally in G^{ϵ}_{ρ} it follows that $\alpha_i\alpha_j^{-1} \in \langle\alpha_1, \ldots, \alpha_{i-1}\rangle$. But then $\alpha_i = (\alpha_i\alpha_j^{-1})\alpha_j \in \langle\alpha_1, \ldots, \alpha_{i-1}\rangle$, a contraction.

We can also prove that $|[\alpha, \beta]| < \min\{|\alpha|, |\beta|\}$. Note first that the rotational parts satisfy $|r[\alpha, \beta]| \le 2|r(\alpha)||r(\beta)| \le 2|\alpha||\beta|$. Before considering the translational parts, note that given any $A \in SO(n)$ and $x \in \mathbb{R}^n$ we have $|(Id - A)x| \le 2\sin(|A|/2) \cdot |x| \le |x|$. The commutator therefore satisfies

$$\begin{aligned}
t[\alpha, \beta] &= -ABA^{-1}B^{-1}b - ABA^{-1}a + Ab + a \\
&= A(I - B)A^{-1}a + AB(I - A^{-1})B^{-1}b \\
|t[\alpha, \beta]| &\le 2|a|\sin\left(\frac{1}{2}|B|\right) + 2|b|\sin\left(\frac{1}{2}|A^{-1}|\right) \\
&\le |r(\beta)||t(\alpha)| + |r(\alpha)||t(\beta)| \;\le\; 2\frac{\rho}{\epsilon}|\alpha||\beta|.
\end{aligned}$$

Therefore $[\alpha_i, \alpha_j] \in \langle\alpha_1, \ldots, \alpha_{\min\{i,j\}-1}\rangle$, so that the length of commutators amongst generators of $\langle G^{\epsilon}_{\rho}\rangle$ is bounded by d. Induction on the formula $[\alpha\beta, \gamma] = [\beta, \gamma] \cdot [[\gamma, \beta], \alpha] \cdot [\alpha, \gamma]$ proves that the length of any commutator chain of elements in $\langle G^{\epsilon}_{\rho}\rangle$ is bounded by d. □

6.2 Proof of (ii)

The lemmas suffice to show that G^ϵ for $\epsilon < 1/2$ is actually a translation group. Assume there is some $\gamma \in G$ with $|r(\gamma)| < \frac{1}{2}$. Let $\mathbb{R}^n = E \oplus E^\perp$ be the orthogonal decomposition where E is the subspace of maximal rotational angle of $r(\gamma)$. Given some $x \in \mathbb{R}^n$ let $x = x^E + x^\perp$ denote the corresponding orthogonal vector decomposition.

Let $\delta \in (0, \frac{1}{2})$ (to be chosen later, and will depend on γ alone) and put $\rho = 2|t(\gamma)|$. Pick some $x \in E$ with $|x| = \frac{3}{4}\rho$. The first lemma guarantees some $\alpha \in G^\delta_\rho$ with $|t(\alpha) - x| \leq \frac{1}{4}\rho$. Consequently $|t(\gamma)| \leq |t(\alpha)|$ and $|t(\alpha)| < 2|t(\alpha)^E|$. Let $\alpha_0 = \alpha$ and

$$\alpha_k \;=\; [\alpha_{k-1}, \gamma]$$

be the k-fold iterated commutator. For convenience put $A_k = r(\alpha_k)$, $a_k = t(\alpha_k)$ and $C = r(\gamma)$, $c = t(\gamma)$. From Finsler geometry we have

$$|A_{k+1}| \;=\; |[A_k, C]| \;\leq\; 2|C||A_k| \;<\; |A_k|$$

so that $|A_i| < |A| \leq \delta$. Consider the decomposition

$$\begin{aligned} t(\alpha_{k+1}) \;\triangleq\; a_{k+1} \;&=\; -A_k C A_k^{-1} C^{-1} c \;-\; A_k C A_k^{-1} a_k \;+\; A_k c \;+\; a_k \\ &=\; (Id \,-\, C)\, a_k \;+\; (Id \,-\, [A_k, C])\, C a_k \;+\; A_k C \left(Id \,-\, A_k^{-1}\right) C^{-1} c \\ &=\; (Id \,-\, C)\, a_k \;+\; (Id \,-\, A_{k+1})\, C a_k \;+\; A_k C \left(Id \,-\, A_k^{-1}\right) C^{-1} c. \end{aligned}$$

It is easy to prove that as long as $|B| = \theta \leq \pi$, then $|(Id - B)f| \leq 2\sin(\theta/2)|f| \leq |B||f|$. As a first application we get

$$\begin{aligned} |a_{k+1}| \;&\leq\; |C||a_k| \;+\; |A_{k+1}||a_k| \;+\; |A_k||c| \\ &\leq\; \left(\frac{1}{2} \;+\; \delta\right) |a_k| \;+\; \delta|a|. \end{aligned}$$

Since commutators are at most $d = d(n)$ long, by iterating we get (after choosing δ) that $|a_{k+1}| < |a|$. From this, we can also estimate $|a_{k+1}|$ from below. Using

$$a^E_{k+1} \;=\; (Id \,-\, C)\, a^E_k \;+\; [(Id \,-\, A_{k+1})\, C a_k]^E \;+\; \left[A_k C \left(Id \,-\, A_k^{-1}\right) C^{-1} c\right]^E$$

and another triangle inequality,

$$\begin{aligned} |a^E_{k+1}| \;&\geq\; |\,(Id \,-\, C)\, a^E_k| \,-\, |\,[(Id \,-\, A_{k+1})\, C a_k]^E\,| \,-\, |\,\left[A_k C \left(Id \,-\, A_k^{-1}\right) C^{-1} c\right]^E\,| \\ &\geq\; |\,(Id \,-\, C)\, a^E_k| \,-\, |\,(Id \,-\, A_{k+1})\, a_k| \,-\, |A_k C \left(Id \,-\, A_k^{-1}\right) C^{-1} c| \\ &\geq\; 2|a^E_{d-1}| \sin\left(\frac{\theta}{2}\right) \,-\, |A_d||a_{d-1}| \,-\, |A_{d-1}||c| \\ &\geq\; 2|a^E_{d-1}| \sin\left(\frac{\theta}{2}\right) \,-\, 2\delta|a| \end{aligned}$$

Choosing δ small enough and iterating this inequality d times, recalling that $|a^E_0| > \frac{1}{2}|a_0| > \frac{1}{2}|c|$, we get that $|a^E_d| > 0$. This contradicts the fact that d-fold commutators vanish. □

6.3 Proof of (i)

To prove that the translation subgroup Γ is normal, let $\alpha \in G$ and $\beta \in \Gamma$ be the transformations $\alpha(x) = Ax + a$ and $\beta(x) = x + b$. Then

$$\begin{aligned} r(\alpha\beta\alpha^{-1}) &= I \\ t(\alpha\beta\alpha^{-1}) &= Ab. \end{aligned}$$

so conjugation fixes Γ. (This also follows from the fact that $\mathbb{R}^n \lhd SO(n) \ltimes \mathbb{R}^n$ and $\Gamma \subset \mathbb{R}^n$.)

Finally note that the rotational parts of any two elements in G/Γ are separated by at least $\frac{1}{2}$. For if $\alpha, \beta \in G$ have $|r(\alpha)r(\beta)^{-1}| < \frac{1}{2}$ then $|AB^{-1}| = 0$ is translational, so in particular $r(\alpha) = r(\beta)$, and $\alpha \equiv \beta \,(mod\, \Gamma)$. Thus elements of G/Γ can be mapped injectively onto a set of $\frac{1}{2}$-separated points on $SO(n)$. Therefore G/Γ is uniformly bounded in terms of n.

Chapter 7

Gromov's almost flat manifold theorem I

February 23, 2010

Let $D = D(M)$ is the diameter of M and $K = \max_{p \in M} |sec_p|$ is the largest sectional curvature that appears on M. Gromov proves the following:

Theorem 7.0.1 (Gromov's almost flat manifold theorem) *There is an $\epsilon > 0$ so that if M is a compact Riemannian manifold and $D^2 K < \epsilon$, then $\pi_1(M)$ has a nilpotent subgroup Γ of finite index, and M is a finite quotient of a nilmanifold.*

7.1 Short loops, short relations, and the Gromov product

The condition $D^2 K \leq \epsilon$ is scale-invariant, so we work in the scale $D(M) = 1$. Making ϵ smaller enforces tighter bounds on K. The maximal rank radius r_{max} of the exponential map, which satisfies $r_{max} \geq \pi/\sqrt{K} \geq \pi\epsilon^{-1/2}D$, can be made arbitrarily large compared to the diameter. We can, somewhat informally, regard exp as a kind of large covering map for M, and most of our work will be done in a large ball in this space.

A *short loop* will be a geodesic from the basepoint to itself, of length shorter than the rank radius. A *short homotopy* is a homotopy through curves that are shorter than the rank radius. Given two short loops a, b, the usual path-product can be deformed, through a homotopy that keeps the endpoints fixed, to a geodesic loop. If this deformation can be done through a short homotopoy, then the final path, denoted $\beta * \alpha$ is called the *product between short loops a and b* (also called “Gromov’s product”), and is unique, by Klingenberg’s lemma. If $|\alpha| + |\beta| < r_{max}$ then $\beta * \alpha$ is defined. If the deformation cannot be done through a short homotopy, then $\beta * \alpha$ is not defined.

Denote by Γ the set of short loops. Clearly $\alpha \in \Gamma$ implies $\alpha^{-1} \in \Gamma$. Associativity holds when the sum of lengths of the paths is less than the rank radius: $\gamma * (\beta * \alpha) = (\gamma * \beta) * \alpha$ if $|\gamma| + |\beta| + |\alpha| < r_{max}$. Thus Γ is (essentially) a pseudogroup.

It is fairly easy to show that Γ possesses all generators and relations that exist in $\pi_1(M)$. To see that generators exist in Γ, note that any element of $\pi_1(M)$ can be expressed as a product of elements of length $\leq 2D(M)$ (this can be seen, for instance, by working in the universal covering: if a representative path there crosses fundamental domain boundaries, is can be deformed into a series of paths, the interior of each of which lies in a single fundamental domain).

It is slightly more difficult to see that the relations are all present in Γ as well (as long as say $\epsilon^{1/2} < \pi/5$).

7.2 Homotopy errors

One would like the Gromov product of short loops to commute with composition of holonomy actions. But the Gromov product involves a homotopy, so this cannot be true in general. However there is a homotopy approximation theorem. Let $\gamma, \tilde{\gamma} : [0,1] \to M$ be homotopic paths from p to q, and let $\triangle$ denote the homotopy between them. Let $|\triangle|$ the area of the homotopy, and let $K(\triangle)$ denote the largest sectional curvature at any point of $\triangle$, and let L denote the length of the longest path in the homotopy. Now let $X, \tilde{X}$ solve $\nabla_{\dot{\gamma}} X = 0$, $\nabla_{\dot{\tilde{\gamma}}} \tilde{X} = 0$ with $X(0) = \tilde{X}(0)$, and let $Y, \tilde{Y}$ solve $\nabla_{\dot{\gamma}} Y = 0$, $\nabla_{\dot{\tilde{\gamma}}} \tilde{Y} = 0$ with $Y(0) = \tilde{Y}(0) = 0$. Then

$$\begin{aligned} |X(1) \,-\, \tilde{X}(1)| &\leq \frac{4}{3} \cdot K(\triangle) \cdot |\triangle| \\ |Y(1) \,-\, \tilde{Y}(1)| &\leq \frac{4}{3} \cdot K(\triangle) \cdot L \cdot |\triangle|. \end{aligned}$$

This allows an estimation of the difference between the rotational and translational parts of the Gromov product of two loops and the standard loop product. Considering the triangle $\triangle$ (in T_pM) formed by α, β, and $\beta * \alpha$, we get (say) $|\triangle| \leq \frac{3}{2} \frac{|t\alpha||t\beta|}{2}$. Then

$$d\left(r(\beta * \alpha),\, r(\beta) r(\alpha)\right) \;\leq\; K|t(\alpha)||t(\beta)|.$$

If L denotes the length of the largest path in the homotopy then $L \leq |t\alpha| + |t\beta|$, so

$$\left|t(m(\beta)m(\alpha)) \,-\, t(\beta * \alpha)\right| \;=\; \left|r(\beta)t(\alpha) + t(\beta) \,-\, t(\beta * \alpha)\right| \;\leq\; K\left(|t(\alpha)| + |t(\beta)|\right)|t(\alpha)||t(\beta)|.$$

For commutators the situation is perhaps better than expected:

- $d\left(r[\beta,\alpha],\,[r\beta,r\alpha]\right) \;\le\; \frac{4}{3}K\left(|t\alpha||t\beta| \,+\, \frac{1}{2}|t[\beta,\alpha]|\left(|t\alpha|+|t\beta|\right)\right)$
- $|t[\beta,\alpha] \,-\, t[m\alpha,m\beta]| \;\le\; \frac{4}{3}K\left(|t\beta|+|t\alpha|\right)\left(|t\alpha||t\beta| \,+\, \frac{1}{2}|t[\beta,\alpha]|\left(|t\alpha|+|t\beta|\right)\right)$
- $\sqrt{K}|t[\beta,\alpha]| \;\le\; |r\alpha|\sinh\left(\sqrt{K}|t\beta|\right) + |r\beta|\sinh\left(\sqrt{K}|t\alpha|\right) + \frac{2}{3}K|t\alpha||t\beta|\sinh\left(\sqrt{K}\left(|t\alpha|+|t\beta|\right)\right)$

Recall that for rotations $A, B \in \mathcal{O}(n)$ we have $|[A,B]| \le 2|A||B|$. For Euclidean motions α, use $|\alpha| = \max\{|r(\alpha)|,\, 3\sqrt{K}|t(\alpha)|\}$.

Thus if $|\alpha|, |\beta| \le \frac{1}{3}$, then

$$\begin{aligned} &|t(\alpha)|, |t(\beta)| \;<\; \frac{1}{9K^{\frac{1}{2}}} \\ &|m[\beta,\alpha]| \le 2.4|m\alpha||m\beta| \le 0.8\min\{|m\alpha|, |m\beta|\}. \end{aligned}$$

7.3 Commutator length

Gromov's striking application is that the commutator length of the subgroup of $\pi_1(M)$ generated by elements of rotation $< \frac{1}{3}$ is bounded in terms of the dimension n only.

Proposition 7.3.1 *Given $\delta < \frac{1}{3}$, let $\widetilde{\Gamma}^\delta$ be some collection of short loops α with $r\alpha < \delta$. Then the group $\left\langle \widetilde{\Gamma}^\delta \right\rangle \subset \pi_1(M)$ is nilpotent with degree of nilpotency bounded by a dimensional constant $d = d(n)$.*

<u>*Pf*</u> Let $\widetilde{\Gamma}^\delta$ be any set of short loops α with $r\alpha < \delta$. We choose a short basis: pick $\alpha_1 \in \widetilde{\Gamma}^\delta$ with minimal $|\alpha_1|$. Pick $\alpha_{i+1} \in \widetilde{\Gamma}^\delta$ with $|m\alpha_{i+1}|$ minimal in $\widetilde{\Gamma}^\delta - \langle \alpha_1, \ldots, \alpha_i \rangle$. It is not necessary to prove that this process stops.

We can prove that $|m(\alpha_i\alpha_j^{-1})| \ge \max\{|m\alpha_i|, |m\alpha_j|\}$. For if $|m(\alpha_i\alpha_j^{-1})| < \max\{|\alpha_i|, |\alpha_j|\}$ (wlog $|\alpha_i| < |\alpha_j|$) then $\alpha_i\alpha_j^{-1} \in \langle \alpha_1, \ldots, \alpha_{i-1} \rangle$ and so too $\alpha_i = \left(\alpha_i\alpha_j^{-1}\right)\alpha_j \in \langle \alpha_1, \ldots, \alpha_{i-1} \rangle$, a contradiction, so therefore the short basis has $|m(\alpha_i, \alpha_j^{-1})| \ge \max\{|m\alpha_i|,\, |m\alpha_j|\}$. After accounting for the homotopy error, we get, say,

$$|m(\alpha_i)m(\alpha_j^{-1})| \;\ge\; \max\left(|m(\alpha_i)| - \frac{1}{27}|m(\alpha_j)|,\; |m(\alpha_j)| - \frac{1}{27}|m(\alpha_i)|\right).$$

Our lemma from the Finsler geometry of $\mathcal{SO}(n)$, still goes through, and there is a uniform bound on d. We have already proved that $|m[\alpha,\beta]| < \min\{|m\alpha|, |m\beta|\}$, so for elements $\alpha, \beta \in \widetilde{\Gamma}^\delta$, it follows that $[\alpha_i, \alpha_j] \in \left\langle \alpha_1, \ldots, \alpha_{\min\{i,j\}=1} \right\rangle$ so that commutators of basis elements have length $\le d$. An induction on $[\alpha\beta, \gamma] = [\beta,\gamma][[\gamma,\beta],\alpha][\alpha,\gamma]$ proves that any commutator in $\langle \alpha_1, \ldots, \alpha_d \rangle$ has length $\le d$. □

7.4 Density of subgroups of small rotational parts

Let Γ^δ be the set of short loops α with $r\alpha < \delta$, let Γ_ρ be the set of short loops α with $t\alpha < \rho$, and put $\Gamma^\delta_\rho = \Gamma^\delta \cap \Gamma_\rho$.

We have found pseudogroups Γ^δ that generate nilpotent subgroups of $\pi_1(M)$. But so what? It is not even clear that such pseudogroups are nontrivial, let alone generators of finite-index subgroups of $\pi_1(M)$. Gromov has a Dirichlet-principle type argument that says that $\langle \Gamma^\delta_\rho \rangle \subset \pi_1(M)$ has 'lots of elements'. Enough, in fact, to generate finite-index subgroups of $\pi_1(M)$.

Lemma 7.4.1 *Given $\delta < \frac{1}{3}$ and $R < \infty$, there is an $\epsilon > 0$ and a $\rho > R$ so that if $DK^2 < \epsilon$, then the set of translational parts of elements in Γ^δ_ρ is $\rho/4$-dense in the ball of radius $3\rho/4$.*

Pf

Pick $\rho_i = 20^i(D(M) + R)$. Assuming the lemma is false, pick $x_i \in B_{3\rho_i/4}$ so that $\mathrm{dist}(x_i, \Gamma^\delta) \geq \rho_i/4$. Since the translation parts of $\pi_1(M)$ is $D(M)$-dense in ρ_i, we can find $\alpha_i \in \pi_1(M)$ with $|t\alpha_i - x_i| < \rho_i/4$, though by assumption $r\alpha_i \geq \delta$.

Fixing i we can show that the rotation parts $r\alpha_i$ are all δ-separated. For convenience put $A_i = r\alpha_i$ and $a_i = t\alpha_i$, so that $m\alpha_i : T_pM \to T_pM$ is given by $X \mapsto A_iX + a_i$. Note that

$$\begin{aligned}
|t(\alpha_i * \alpha_j^{-1})| &\leq |-A_iA_j^{-1}a_j| + |a_i - x_i| + |x_i| + \mathit{error} \\
&\leq \rho_j + D + \frac{\rho_i}{4} + \mathit{error} < \rho_i \\
|t(\alpha_i * \alpha_j^{-1}) - x_i| &= |-A_iA_j^{-1}a_j + a_i - x_i| + \mathit{error} \\
&\leq |a_j| + |a_i - x_i| \leq \rho_j + d + \mathit{error} < \rho_i/4.
\end{aligned}$$

The 'error' is controlled by $\frac{1}{2}\min\{|a_i|, |a_j|\}$. The second inequality implies $\alpha_i * \alpha_j^{-1} \notin \Gamma^\delta$, but the first inequality has $|\alpha_i * \alpha_j^{-1}| < \rho_i$, meaning $r(\alpha_i * \alpha_j^{-1}) \geq \delta$, so $r\alpha_i$ is δ-separated from each of the $r\alpha_1, \ldots, r\alpha_{i-1}$.

For each δ, there is a uniformly finite number of elements of $\mathcal{SO}(n)$ that can be δ-separated, so, as long as ρ_d is within the maximal rank radius, the length of the list $\alpha_1, \ldots, \alpha_d$ is bounded in terms of n. □

Chapter 8

Gromov's almost flat manifold theorem II

February 25, 2010

8.1 Small rotation implies almost-translation

Lemma 8.1.1 *Given any small number $\eta > 0$, there are numbers $\epsilon = \epsilon(n, \delta, \eta, R)$, $\rho = \rho(\delta, \eta) > R$ so that $\gamma \in \Gamma_\rho^{1/3}$ implies $\gamma \in \Gamma_\rho^\eta$, provided M is ϵ-flat.*

<u>*Pf*</u>

Assume there is a motion γ with $r\gamma < \frac{1}{3}$ and $t\gamma < \rho$, where ρ will be chosen momentarily. Put $C = r\gamma$ and $c = t\gamma$, so the affine holonomy action $m\gamma : T_pM \to T_pM$ is $X \mapsto CX + c$. Write $\mathbb{R}^n = E + E^\perp$ where E is the plane of maximum rotation of C. Pick $x \in E$ with $|x| = 2|c|$. There is some $\rho > 2|c|$ so that $t\left(\Gamma_\rho^\delta\right)$ is $\rho/4$-dense in $B(3\rho/4)$, so there is a short loop α with $|t\alpha - x| < \rho/4$ and $r\alpha < \delta$. Note that $|t\gamma| < |t\alpha|$. Put $\alpha_0 = \alpha$ and $\alpha_k = [\alpha_{k-1}, \gamma]$. For convenience write $A_k = r\alpha_k$ and $a_k = t\alpha_k$. Note that

$$\begin{aligned}
|A_{k+1}| &= |r[\alpha_k, \gamma]| \\
&\leq K\left(2|t\alpha_k||t\gamma| + |t[\alpha_k, \gamma]|\left(|t\alpha_k| + |t\gamma|\right)\right) + |[A_k, C]| \\
&\leq K\left(2|t\alpha_k||t\gamma| + |t[\alpha_k, \gamma]|\left(|t\alpha_k| + |t\gamma|\right)\right) + 2|C||A_k|
\end{aligned}$$

Since $k+1 < d$ (the constant $d = d(n)$ is the nilpotency degree from the previous lemma), we can choose ϵ (therefore K) so small, that after iterating at most d times, we get $|A_k| <$

$|A| < \delta$. Now we can estimate the translation parts of α_k from above

$$\begin{aligned} a_{k+1} &= -A_k C A_k^{-1} C^{-1} c \,-\, A_k C A_k^{-1} a_k \,+\, A_k c \,+\, a_k \,+\, error \\ &= (Id \,-\, C)\, a_k \,+\, (Id \,-\, A_{k+1})\, C a_k \,+\, A_k C \left(Id \,-\, A_k^{-1}\right) C^{-1} c \,+\, error. \end{aligned}$$

Thus

$$\begin{aligned} |a_{k+1}| &\leq |C||a_k| \,+\, |A_{k+1}||a_k| \,+\, |A_k||c| \,+\, error \\ &= \left(\frac{1}{3} \,+\, \delta\right) |a_k| \,+\, \delta |c| \,+\, error \\ &< \left(\frac{1}{3} \,+\, \delta\right) |a_k| \,+\, \delta |a| \,+\, error. \end{aligned}$$

Iterating, we get $|a_{k+1}| < |a|$. From this we can estimate $|a_{k+1}^E|$ from below.

$$\begin{aligned} |a_{k+1}^E| &\geq |C||a_k^E| \,-\, |A_{k+1}||a_k| \,-\, |A_k||c| \,-\, error \\ &\geq |C||a_k^E| \,-\, 2\delta |a| \,-\, error. \end{aligned}$$

If $|C|$ is large enough compared to δ and ϵ (which controls the error), then $|a_d^E| > 0$ after iterating $d = d(n)$ times, an impossibility. Therefore $|C|^d$ is small compared to ϵ. □

Note that, for an appropriately large ρ and small ϵ, the set $\Gamma_\rho^{1/3}$ is a pseudogroup under the Gromov product. This is because rotational parts cannot build up under the product: as soon as they always remain smaller than η. Note also that because at least $\rho/|t(\alpha)|$ iterations of α still lie in Γ_δ^η, we have

$$\alpha \in \Gamma_\eta^\delta \quad \Longrightarrow \quad |r(\alpha)| \;\leq\; \eta \frac{|t(\alpha)|}{\rho}.$$

Finally note that translational errors for products in $\Gamma_\rho^{1/3}$ are extremely tiny:

$$\begin{aligned} &|t(\beta * \alpha) \,-\, t(\alpha) \,-\, t(\beta)| \;\leq\; \eta \frac{|t(\alpha)||t(\beta)|}{\rho}(1+\epsilon) \\ &|t([\beta, \alpha])| \;\leq\; 2\eta \frac{|t(\alpha)||t(\beta)|}{\rho}(1+\epsilon). \end{aligned}$$

8.2 Gromov's Normal Basis

We have found a normal subgroup of finite index in $\pi_1(M)$, but we have not found that the corresponding cover is a Lie group. The first step in finding a Lie group is constructing a "Gromov normal basis".

Let $\delta_1 \in \Gamma_\rho^\eta$ be a minimal element. By the commutator estimates, $[\delta_1, \Gamma_\rho^\delta] = 0$. Given any $\alpha \in \Gamma_\rho^\eta$ let $\alpha_i = \delta_1^i * \alpha$ whenever the product is in Γ_ρ. Since Gromov products among elements in Γ_ρ^η are almost-translations, we can find a unique $\alpha_j \in \{\alpha_i\}$ with

$$\begin{aligned} \langle t(\alpha_j), \delta_1 \rangle &\geq 0 \\ \langle t(\tilde{\alpha}_{j-1}), \delta_1 \rangle &< 0. \end{aligned}$$

Put $\tilde{\alpha} = \alpha_j$. Finally let α' be the projection of $\tilde{\alpha}$ onto the subspace orthogonal to $t(\delta_1)$ in $\mathbb{R}^n = T_p(M)$. Define the product $\beta' * \alpha'$ to be the projection of the product $\tilde{\beta} * \tilde{\alpha}$ onto the compliment. The collection of the α' with this product form a pseudogroup Γ'.

Using that $|t(\delta_1)|$ is smaller than $|t(\alpha)|$ we can prove $|t(\alpha')| \leq |t(\tilde{\alpha})| \leq 1.5|t(\alpha')|$, and therefore the translation estimates from above still apply to products in Γ'.

Perform the process again: there is some $\delta_2' \in \Gamma'$ (projection of some $\tilde{\delta}_2$) with shortest translation part among elements of Γ', etc.

Due to the denseness of Γ_ρ^η in $\mathbb{R}^n$, this process will terminate with exactly n elements $\delta_1, \tilde{\delta}_2, \tilde{\tilde{\delta}}_2, \ldots$. The subgroup of Γ_ρ^η generated by these n elements is actually Γ_ρ^η itself.

This is because any element of Γ_ρ^η is first translated by δ_1 so it is 'almost' in the complimentary plane. Then is is translated by δ_2 so it is 'almost' in the plane complimentary to δ_1, δ_2, etc, until it is translated until it is 'almost' at the origin. But this contradicts the minimality of the n^{th} element δ_n.

Let G_k indicate the subpseudogroup generated by $\delta_1, \ldots, \delta_k$. The commutator estimates indicate that $[\delta_i, \delta_j]$ is mush smaller than δ_i or δ_j. Its representative $\widetilde{[\delta_i, \delta_j]}$ will also be much smaller. But since δ_i, δ_j were chosen minimally, this implies $[\delta_i, \delta_j] \subset G_{\min\{i,j\}-1}$.

Therefore each element in Γ_ρ^η has a unique expression in the form $\delta_1^{k_1} * \cdots * \delta_n^{k_n}$.

8.3 Construction of the Lie group and the covering map

According to a theorem of Malcev, the product of two elements of Γ^η is

$$\delta_1^{k_1} * \cdots * \delta_n^{k_n} \; * \; \delta_1^{l_1} * \cdots * \delta_n^{l_n} \; = \; \delta_1^{P_1} * \cdots * \delta_n^{P_n},$$

where the P_i are polynomials of degree $\leq n + 1 - i$ in the $k_1, \ldots, k_n, l_1, \ldots, l_n$, called the Malcev polnomials.

Now we can identify $p = \delta_1^{k_1} * \cdots * \delta_n^{k_n}$ with the lattice point $\sum k_i \delta_i$, and use the Malcev polynomials to determine a product on this lattice. Replacing the integers k_i with real numbers, and still using the polynomials to determine products, we now have a nilpotent group structure on $\mathbb{R}^n$.

Through the exponential map, large balls around the origin in $\mathbb{R}^n$ can be identified with large balls around a basepoint in the universal cover $\tilde{M}$ of M. Now Γ^η acts by left translation on $\mathbb{R}^n$ and also by deck transformation on $\tilde{M}$. These actions are almost compatible, so after an appropriate center-of-mass averaging, we get a Γ^η-equivariant map from the Lie group to $\tilde{M}$.

Chapter 9

Fukaya's Theorem

March 2, 2010

9.1 Statement

Theorem 9.1.1 (Fukaya) *Given n, $\mu > 0$, there is a number ϵ so that whenever N^n, M are Riemannian manifolds with $|sec| \leq 1$, $\mathrm{inj}(N) > \mu$, and $d_{GH}(N, M) < \epsilon$, then there is a (differentiable?) submersion $f : M \to N$ so that (M, N, f) is a fiber bundle, the fibers are quotients of nilmanifolds, and $e^{-\tau(\epsilon)} < |df(\xi)|/|\xi| < e^{\tau(\epsilon)}$.*

We use τ to indicate a function of ϵ with $\lim_{\epsilon \to 0} \tau(\epsilon) = 0$. We set up some notation that will be used throughout.

$$\begin{aligned} R &= \min\{\mu, 1\}/2 \\ \sigma &= \text{a small number}, \quad 0 < \epsilon << \sigma << 1 \\ r &= \sigma R \end{aligned}$$

9.2 Embedding into an l^2 space

Let (Z, d) be a discrete metric space, with ϵ-almost isometries into M and N, $j_M : Z \to M$ and $j_N : Z \to N$. Since $d_{GH}(M, N) < \epsilon$, we can choose (Z, d) and j_M, j_N so that $Z = (z_1, \dots)$ is a countable set, M (resp. N) is in the ϵ-neighborhood of $j_N(Z)$ (resp. $j_M(z)$), and so that $j_M(Z)$ (resp. $J_N(z)$) is ϵ-dense and $\epsilon/4$-separated in M (resp. N).

Consider the space $\mathbb{R}^Z = l^2(Z)$, the Hilbert space on Z. If ϵ is small compared to μ then we can define $f_N : N \to \mathbb{R}^Z$ by setting

$$p \mapsto (\mathrm{dist}_N(p, z_1), \dots) .$$

This map is 1-1, but not differentiable since $\mathrm{dist}_N(z_i, \cdot)$ is Lipschitz and not C^1 (also it is not a map into $l^2(Z)$ unless $\#\{Z\} < \infty$). However we can compose this with a C^∞ cutoff function $h : \mathbb{R} \to \mathbb{R}$ that is constant at 0, and equals zero outside a definite radius. Specifically,

$$\begin{array}{rcl} h(t) & = & 1 \quad \text{if } t \le 0 \\ h(t) & = & 0 \quad \text{if } t \ge r \\ h'(t) & \in & [-\kappa/r, 0) \quad \text{if } t \in (0, r/8] \cup [7r/8, r) \\ h'(t) & \in & [-\kappa/r, -2/r] \quad \text{if } t \in (r/8, 7r/8). \end{array}$$

Now define

$$f_N(p) \ = \ (h(\mathrm{dist}_N(p, z_1)), \dots) .$$

Let

$$K = \sup_{x \in N} \# \left(B_r(x) \cap j_N(Z) \right) .$$

The following hold, for appropriate constants C, C_1, C_2:

- f_N is an embedding
- $\exp^\perp : T^\perp N :\to \mathbb{R}^Z$ is a diffeomorphism out to radius $C\sqrt{K}$.
- (quasi-isometry) we have $|df_N(\xi)|/|\xi| \in (C_1\sqrt{K}, C_2\sqrt{K})$
- If $d_N(x, y)$ is small enough compared to ϵ, σ, and μ, then

$$d(x, y) \le C K^{-1/2} \operatorname*{dist}_{\mathbb{R}^Z}(f_N(x), f_N(y)).$$

For a proof see A. Katsuda, *Gromov's convergence theorem and its applications* (1984). We would like to say something about a similar map $M \to \mathbb{R}^Z$, but we cannot expect the distance functions $\mathrm{dist}_M(z_i, \cdot)$ can themselves ever be made differentiable. Yet we can smooth them. For $p \in M$ set

$$d_z(p) \ = \ \fint_{B_\epsilon(z)} \mathrm{dist}_M(p, y)\, dy.$$

Then d_z is C^1 (but not C^2), for if $\xi \in T_pM$ then

$$\xi(d_z)(p) \ = \ \fint_{B_\epsilon(z)} \xi(\mathrm{dist}_M(p, y))\, dy,$$

and $\xi(\mathrm{dist}_M(p, y))$ is defined almost everywhere.

Proposition 9.2.1 *The maps $j_N : N \to \mathbb{R}^Z$, $j_M : M \to \mathbb{R}^Z$ are embeddings, and $j_M(M)$ is in the $6\epsilon\sqrt{K}$-tubular neighborhood of $j_N(N)$.*

<u>*Pf*</u>

We prove the last statement. Since M and N are ϵ-close in the Gromov-Hausdorff sense, we can find a distance function d on $M \coprod N$ that restricts to the Riemannian distance on M and N respectively, and so that M is in the ϵ-neighborhood of N and vice-versa, and with $\mathrm{dist}(j_M(z_i), j_N(z_i)) < \epsilon$. Let p be any point of M and let $p' \in N$ be a point with $d(p, p') < \epsilon$. Then

$$\begin{aligned} d(p, j_M(z_i)) &\leq d(p', j_N(z_i)) + d(p, p') + d(j_N(z_i), j_M(z_i)) \\ d(p', j_N(z_i)) &\leq d(p, j_M(z_i)) + d(p, p') + d(j_N(z_i), j_M(z_i)) \end{aligned}$$

so that

$$|\mathrm{dist}_M(p, j_M(z_i)) \,-\, \mathrm{dist}_N(p', j_N(z_i))| \,\leq\, 2\epsilon.$$

Since $|h'(t)| \leq 2$ we have

$$|h(\mathrm{dist}_M(p, j_M(z_i))) \,-\, h(\mathrm{dist}_N(p', j_N(z_i)))| \,\leq\, 4\epsilon.$$

Then

$$\begin{aligned} |f_M(p) \,-\, f_N(p')|^2 &= \sum_i \left(h(\mathrm{dist}_M(p, j_M(z_i))) \,-\, h(\mathrm{dist}_N(p', j_N(z_i)))\right)^2 \\ &\leq 16K\epsilon^2. \end{aligned}$$

Using the averaged quantity $d_{z_i}(p)$ in place of $\mathrm{dist}_N(z_i, p)$ changes the estimates by at most 2ϵ, so we get the result. □

Now we have a map $f : M \to N$ given by

$$f \,=\, f_N^{-1} \circ \pi \circ \exp^{\perp -1} \circ f_M$$

where π indicates the projection from the normal bundle of $f_N(N)$ onto N.

9.3 $f : M \to N$ is a fiber bundle

We have to prove that $f_M(M)$ is transverse to the fibers of the normal bundle of $f_N(N)$ in $\mathbb{R}^Z$. This follows directly from the following proposition.

Proposition 9.3.1 *Given any $\nu > 0$, one can choose ϵ, σ so that the following holds. If $p \in M$ and $p' = f(p)$, then given any $\xi' \in T_{p'}N$ there exists a $\xi \in T_pM$ such that*

$$\frac{|df_M(\xi) \,-\, df_N(\xi)|}{|df_N(\xi)|} \,\leq\, \nu.$$

Pf

Let $l' : [0, t']$ be a unit-speed geodesic in N with $l'(0) = p'$ and $\frac{Dl'}{dt} = \xi'$. Let $l : [0, t]$ be a geodesic in M with $l(0) = p$ and $\mathrm{dist}_{\mathbb{R}^Z}(l(t), l'(t')) < \epsilon$. Now let l_i' be a geodesic from $j_N(z_i)$ to p' and let l_i be a geodesic from a point $y \in B_\epsilon(j_M(z_i))$ to p. Let θ_i be the angle between l and l_i, and let θ_i' be the angle l' and l_i'. We prove that

$$\left| \frac{d}{dt}\Big|_{t=0} h(\mathrm{dist}_N(y,p)) \ - \ \frac{d}{dt}\Big|_{t=0} h(\mathrm{dist}_N(j_N(z_i),p')) \right| \ \leq \ \nu.$$

We break the proof into two parts; when $\mathrm{dist}(j_N(z_i), p) < r/8 - \epsilon$ or $\mathrm{dist}(j_N(z_i), p) > 7r/8 + \epsilon$, and when $\mathrm{dist}(j_N(z_i), p) \in [r/8 - \epsilon, 7r/8 + \epsilon]$.

In the first case,

$$\left| \frac{d}{dt} h(\mathrm{dist}_M(y,p)) \right| \ = \ h' \cdot \mathrm{dist} \ \leq \ \kappa r/8$$
$$\left| \frac{d}{dt} h(\mathrm{dist}_N(j_N(z_i),p')) \right| \ = \ h' \cdot \mathrm{dist} \ \leq \ \kappa r/8$$

so that

$$\left| \frac{d}{dt} h(\mathrm{dist}_N(y,p)) \ - \ \frac{d}{dt} h(\mathrm{dist}_N(j_N(z_i),p')) \right| \ \leq \ 2\kappa r/8 < \kappa\sigma/8$$

Now we consider the second case. By the first variation formula, we have to prove that $|\theta_i - \theta_i'|$ is small. By Toponogov's comparison theorem, we have to prove the following.

Lemma 9.3.2 *Given $\delta > 0$, $\mu > 0$, there is a ν with the following properties. Given $\delta R < t_1, t_2 < R$, assume $l_1 : [0, t_1] \to M$, $l_2 : [0, t_2] \to M$ are geodesics with $l_1(0) = l_2(0) = p$, and $l_1' : [0, t_1'] \to N$, $l_2' : [0, t_2'] \to N$ are minimal geodesics with $l_1'(0) = l_2'(0) = p'$ with $d(l_1'(t_1'), l_1(t_1)) < \nu$, $d(l_2'(t_2'), l_2(t_2)) < \nu$. If θ and θ' are the angles formed by $l_1(0)$, $l_2(0)$ and $l_1'(0)$, $l_2'(0)$ respectively, then $|\theta - \theta'| < \mu$.*

Pf

□

□

Chapter 10

F-structures I

March 9, 2010

10.1 Partial Actions

Def A *partial action*, A, of a topological group G on a Hausdorff space X is given by

i. The *domain* of the action: a neighborhood $\mathcal{D} \subset G \times X$ of $\{e\} \times X$.

ii. A continuous map $A : \mathcal{D} \to X$, also written $(g, x) \to gx$, such that $(g_1 g_2)x = g_1(g_2 x)$ whenever $(g_1 g_2, x)$ and $(g_1, g_2 x)$ lie in $\mathcal{D}$.

To emphasize the domain, a partial action A can be written $(A, \mathcal{D})$.

We can form an equivalence relation on the set of partial actions. Two partial actions $(A_1, \mathcal{D}_1)$ and $(A_2, \mathcal{D}_2)$ are equivalent if for any subset $\mathcal{D} \subset \mathcal{D}_1 \cap \mathcal{D}_2$, we have $A_1|_{\mathcal{D}} = A_2|_{\mathcal{D}}$; we will denote an equivalence class by $[A]$. Any global action defines a local action; an equivalence class which has such a member will be called *complete*. Notice that if G is connected, any two global actions in the same equivalence class are identical. An equivalence class of partial actions is called a *local action*.

In the smooth category, the class of local actions of a Lie group G is just the class of homomorphisms from the Lie algebra of G to the Lie algebra of vector fields on X. The completeness of an action is the same as the *global* integrability of the individual vector fields.

A subset $X_0 \subset X$ is called $[A]$-invariant if whenever $x_0 \in X_0$ and $(g, x_0) \in \mathcal{D}$ (for some $\mathcal{D}$ associated to a partial action $A \in [A]$) then $gx_0 \in X_0$. The intersection of $[A]$-invariant sets is $[A]$-invariant, so any point x lies in a minimal $[A]$-invariant set, called the orbit of x, denoted $\mathcal{O}_x$ or just $\mathcal{O}$. The orbits partition the space X.

A local action $[A]$ on X can be restricted to any subset $U \subset X$ by restricting the domain $\mathcal{D}$ of any representative of $[A]$ to any open subset $\mathcal{D}'$ that contains $\{e\} \times U$ and which obeys (*ii*) from above. If $(A_1, \mathcal{D}_1)$ represents a local action on U_1 and $(A_2, \mathcal{D}_2)$ represents a local action on U_2 and if $A_1|_{\mathcal{D}_1 \cap \mathcal{D}_2} = A_2|_{\mathcal{D}_1 \cap \mathcal{D}_2}$, a partial action on $U_1 \cup U_2$ is can be constructed with domain $\mathcal{D}_1 \cup \mathcal{D}_2$. This of course defined a local action on $U_1 \cup U_2$. Unlike an action, a local action $[A]$ on X pulls back along any local homeomorphism $f : Y \to X$ to a local action $f^*[A]$ on Y.

10.2 $\tilde{g}$-structures and F-structures

Def A *sheaf*, $\mathfrak{F}$, on a topological space X is an association between open sets $U \subset X$ and groups that satisfies the following three axioms.

a) $\mathfrak{F}(U)$ is a group whenever U is an open subset of X

b) If $V \subseteq U$ is an inclusion of open sets, there is a homomorphism (the restriction homomorphism) $\rho_{VU} : \mathfrak{F}(U) \to \mathfrak{F}(V)$ subject to the restrictions that (*i*) $\mathfrak{F}(\varnothing) = \{0\}$, (*ii*) $\rho_{UU} = \mathrm{Id}$, and (*iii*) $W \subseteq V \subseteq U$ implies $\rho_{WU} = \rho_{WV} \circ \rho_{VU}$.

c) If $\{V_\alpha\}$ is an open covering of U and $s_\alpha \in \mathfrak{F}(V_\alpha)$ satisfies $s_\alpha|_{U_\alpha \cap U_\beta} = s_\beta|_{U_\alpha \cap U_\beta}$, then there exists a unique element $s \in \mathfrak{F}(U)$ so that $s|_{U_\alpha} = s_\alpha$.

If $\mathfrak{F}$ only satisfies (*a*) and (*b*) it is called a *presheaf*. The salient feature of sheafs is the *stalk* that exists over each point, and the nature of their global connectedness. Let $\mathfrak{F}$ be a sheaf over M, and let $p \in M$. Let $\{U\alpha\}_{\alpha \in \mathcal{A}}$ be the family of open sets containing p; in fact the $\mathfrak{F}(U_\alpha)$ constitute a directed family of groups. The direct limit is called the stalk at p. A topology can be put on the space of stalks: a neighborhood base is given by the images of the "sections" $\mathfrak{F}(U)$ in the space of stalks. Stalks can be defined if just a presheaf structure exists, and then sections of the space of stalks constitute a sheaf (the *sheafification* of the presheaf).

Let $\mathfrak{F}$ be a sheaf of connected topological groups (note there is some question about topology here; we just accept that there are two topologies, the sheaf topology, and a topology that makes the stalks into Lie groups— for differential geometric applications, usually the sheaf topology is ignored). An *action* of $\mathfrak{F}$ on X is given by a local action of $\mathfrak{F}(U)$ for each open U such that the local actions agree with the sheaf restriction maps. To be explicit, when $x \in V \subset U$ and $g \in \mathfrak{F}(U)$, we have $gx = \rho_{VU}(g)x$ wherever gx and $\rho_{VU}(g)x$ are defined.

A set $S \subset X$ is called invariant if $S \cap U$ is invariant under $\mathfrak{g}(U)$ for all open subsets $U \subset X$. A minimal invariant set is called an orbit. The orbits partition X, and a set that is the disjoint union of orbits is called *saturated*.

We denote the stalk at x by $\mathfrak{F}_x$. If $f : X \to Y$ is a local homeomorphism, we denote by $f^* \mathfrak{F}$ the pullback sheaf.

Def An action of a sheaf $\mathfrak{F}$ is called a *complete local action* if whenever $x \in X$ there exists a neighborhood $V(x)$ of x and a local homeomorphism $\pi : \tilde{V}(x) \to V(x)$ so that $\tilde{V}(x)$ is Hausdorff and

i. If $\tilde{x} \in \pi^{-1}(x)$, then for any open neighborhood $W \subset \tilde{V}(x)$ of x, the structure homomorphism $\mathfrak{F}(W) \to \mathfrak{g}_{\tilde{x}}$ is an isomorphism

ii. The local action of $\pi^* \mathfrak{F}(\tilde{V}(x))$ on $\tilde{V}(x)$ is complete.

If $\pi : \tilde{V}(x) \to V(x)$ is a covering space, the deck transformation group Γ induces a natural action on $\pi^* \mathfrak{F}$, called the *holonomy action*. For $\gamma \in \Gamma$ it is easy to verify $\gamma(gx) = \gamma(g)\gamma(x)$. Specifically, if $g \in \mathfrak{F}(U)$ then $\gamma(g)$ acts on elements $x \in \gamma(U)$ via $\gamma(g) = \gamma \circ g \circ \gamma^{-1}$.

Def A $\tilde{\mathfrak{g}}$-structure $\mathcal{G}$ on X is a sheaf, $\mathcal{G}$, of connected topological groups and a complete local action of $\mathcal{G}$ on X such that the sets $V(x)$ and $\tilde{V}(x)$ can be chosen so that

i. $\pi : \tilde{V}(x) \to V(x)$ is a normal covering map

ii. For all x, $V(x)$ is saturated

iii. For all $\mathcal{O}$, if $x, y \in \overline{\mathcal{O}}$, then $V(x) = V(y)$.

Condition (*iii*) actually implies that $\mathcal{G}$ is a locally constant sheaf on $\overline{\mathcal{O}}$, though not necessarily on neighborhoods of $\overline{\mathcal{O}}$. A $\tilde{\mathfrak{g}}$-structure $\mathcal{G}$ is called *pure* if it is a locally constant sheaf on each $V(x)$.

Def A $\tilde{\mathfrak{g}}$-structure $\mathcal{G}$ is called an F-structure if each stalk $\mathcal{G}_x$ is isomorphic to a torus, and the sets $\tilde{V}(x)$ can be chosen to be finite coverings. If one can choose $\tilde{V}(x) = V(x)$, then $\mathcal{G}$ is called a *T-structure*. If $\tilde{V}(x)$ can be chosen independently of x then $\mathcal{G}$ is called an *elementary F-structure*.

If $\mathcal{G}$ is a $\tilde{\mathfrak{g}}$-structure with sheaf $\mathcal{G}$ and $\mathcal{G}' \subset \mathcal{G}$ is a subsheaf, then, since the action of $\mathcal{G}$ descends to $\mathcal{G}'$, the sheaf $\mathcal{G}'$ comes with a complete local action. This defines a $\tilde{\mathfrak{g}}$-structure $\mathcal{G}'$ called a *substructure*.

Proposition 10.2.1 *If X is a compact manifold that carries an F-structure of positive rank, then $\chi(X) = 0$.*

Pf

On each $\tilde{V}(x)$ a torus acts with no common fixed points, so almost all of its elements have a fixed-point free action. Given such an element with no fixed points, one finds a one-parameter subgroup that acts on $\tilde{V}(x)$, and so $\chi(\tilde{V}(x)) = 0$, so $\chi(V(x)) = 0$. Essentially the same argument shows that $\chi(V(x) \cap V(y)) = 0$. Covering X with finitely many $V(x)$, we get the result. □

Chapter 11

F-structures II - Polarizations and some examples

March 11, 2010

11.1 Atlases and Polarizations

If X is a manifold and for all U, $g(U)$ is a connected topological group and its local action on U is effective, then the restriction maps will be injective. For this kind of $\tilde{\mathfrak{g}}$-structure $\mathcal{G}$, on each $V(x)$ there is a unique pure substructure $\mathcal{G}_\alpha \subset \mathcal{G}\,|_{V(x)}$ with stalk $\mathcal{G}_{\alpha,x} = \mathcal{G}_x$. If every $V(x)$ can be chosen this way, we call $\mathcal{G}$ an effective $\tilde{\mathfrak{g}}$-structure.

We define the *rank* of a $\tilde{\mathfrak{g}}$-structure $\mathcal{G}$ at x to be $\dim \mathcal{O}_x$, and we say $\mathcal{G}$ has positive rank if $\dim \mathcal{O}_x > 0$ for all x.

Def If $\mathcal{G}$ is an effective $\tilde{\mathfrak{g}}$-structure, a collection $\{(U_\alpha, \mathcal{G}_\alpha)\}$ is called an *atlas* for $\mathcal{G}$ if

- the U_α are connected, saturated (w.r.t. $\mathcal{G}$), and open, and form a locally finite covering of X
- each $\mathcal{G}_\alpha \subset \mathcal{G}\,|_{U_\alpha}$ is pure
- given any x, there is an α with $\mathcal{G}_{\alpha,x} = \mathcal{G}_x$.

A *subatlas* $\mathring{A}' \subset \mathring{A}$ is an atlas $\{(U'_\alpha, \mathcal{G}'_\alpha)\}$ so that $U'_\alpha \subset U_\alpha$ and $\mathcal{G}'_\alpha = \mathcal{G}_\alpha\,|_{U'_\alpha}$.

A substructure $\mathcal{P} \subseteq \mathcal{G}$ is called a *polarization* for $\mathcal{G}$ if $\mathcal{P}$ has an atlas so that the rank of $\mathcal{P}_\alpha$ is positive and constant on U_α (the rank of $\mathcal{P}$ may vary with α). A polarization $\mathcal{P}$ is called *pure* if $\mathcal{P}$ is a pure $\tilde{g}$-structure.

Proposition 11.1.1 (regular atlases) *If the F-structure $\mathcal{G}$ on the manifold X (possibly open) has an atlas $\{(U_\alpha, \mathcal{G}_\alpha)\}$, then it has an atlas $\{(\underline{U}_\alpha, \mathcal{G}_\alpha)\}$ for $\mathcal{G}$ with the following properties:*

(1) The sets $\underline{U}_\alpha$ have compact closure

(2) If $x \in \underline{U}_{\alpha_1} \cap \cdots \cap \underline{U}_{\alpha_k}$, then (for some ordering) $\mathcal{G}_{\alpha_1,x} \subseteq \cdots \subseteq \mathcal{G}_{\alpha_k,x}$

(3) Given any $x \in \underline{U}_\alpha$, there is at most one $\underline{U}_\beta$ with $\mathcal{G}_{\alpha,x} = \mathcal{G}_{\beta,x}$. If the manifold is compact or if (1) is dropped, we can assume strict inclusion in (2).

<u>*Pf*</u>

(1) is clear.

(2) We argue inductively. Assume $x \in U_\beta \cap U_\gamma$ but $\mathcal{G}_{\beta,x} \nsubseteq \mathcal{G}_{\gamma,x}$ and $\mathcal{G}_{\gamma,x} \nsubseteq \mathcal{G}_{\beta,x}$. Since $\mathcal{G}_y \neq \mathcal{G}_{\beta,y} \neq \mathcal{G}_{\gamma,y}$ for any $y \in U_\beta \cap U_\gamma$, so that $U_\beta \cap U_\gamma$ is covered by other domains in the atlas. Thus we can replace U_β by $U_\beta - \overline{U_\gamma}$ and U_γ by $U_\gamma - \overline{U_\beta}$, and still retain $X = \bigcup U_\alpha$.

(3) First assume (1) an be dropped or that the manifold is compact. Let $U_1, \ldots, U_k$ be a maximal subcollection so that $\bigcup U_i$ is connected and whenever $x \in U_i \cap U_j$, then $\mathcal{G}_{i,x} = \mathcal{G}_{j,x}$. Set $\underline{U}_1 = U_1$ and let $\underline{U}_2, \ldots, \underline{U}_l$ be the connected components of $\bigcup_i U_i$ where the union is over the U_i the have nonzero intersection with U_1. Now consider the U_i that do not intersect U_1, and repeat this process.

Doing this for all such subcollections, the result follows. □

Proposition 11.1.2 (invariant metrics) *Assume X is a manifold, and let $\mathring{A} = \{(U_\alpha, \mathcal{G}_\alpha)\}$ be a regular atlas for $\mathcal{G}$. If $\mathcal{G}$ has the property that each $\tilde{V}(x) \to V(x)$ is a finite normal covering, then X has a $\mathcal{G}$-invariant metric.*

<u>*Pf*</u>

Let $\mathring{A}' \subset \mathring{A}$. With a partial ordering of the U_α coming from (2) of Proposition 11.1.1, we can choose U_α to be maximal. Cover U'_α by sets $V(x_1), \ldots, V(x_k)$ with $\overline{V(x_i)} \subset U_\alpha$. Put some metric on $V(x_1)$, lift it to $\tilde{V}(x_1)$, and average it over the action of $\mathcal{G}$ and over the deck action. Project back to $V(x_1)$. Put a metric on $V(x_2)$ that agrees with the invariant metric on $V(x_1)$ on the overlap, and perform the same averaging. Eventually this gives an invariant metric on U'_α. This same procedure can be done on some U'_β, only the starting metrics on the $V(x_i)$ must now agree with the metric on U_α where the intersection is nonempty. □

11.2 Examples

Reason for passing to coverings

Let K be the Klein bottle. The torus T acts on the orientable 2-cover of K (which is again the torus). The action of T on the cover gives rise to a local action, which passes back to K. Let $\mathcal{G}$ be the F-structure defined here, with a locally constant sheaf $\mathfrak{g}$. Clearly no local action of $\mathfrak{g}$ on K is complete, but passing to the cover gives a complete action.

Canonical action of a sheaf on its total space

Let $\mathfrak{g}$ be a locally constant sheaf of topological groups over a topological space X, with projection π. Let $\mathfrak{g}^* = \pi^*(\mathfrak{g})$ denote the pullback sheaf. There is a canonical local action of $\mathfrak{g}^*$ on the total space of the sheaf $\mathfrak{g}$. This action is pure, the orbits are just the fibers, and $(\pi^{-1}(X), \mathfrak{g}^*)$ is a pure polarization.

A non-polarized F-structure with a polarization

Consider $\mathbb{S}^3 \subset \mathbb{C}^2$. The Clifford torus, just the set of points $(e^{i\theta_1}, e^{i\theta_2}) \in \mathbb{C}^2$, acts on S^3 via multiplication. Let $\mathcal{G}$ be the F-structure obtained from this action. The structure is pure, though not of constant rank. The natural atlas is just $\{(U_\alpha, \mathcal{G})\}$. It is not a polarization however, since the rank is nonconstant. In fact, there is no polarized atlas for this structure; one must pass to a substructure. Any one-parameter subgroup of the torus besides either of the factors themselves, yields a pure polarized T-structure.

An F-structure with no pure polarization

We consider $\mathbb{S}^3$ as above. If $U \in \mathbb{S}^3$ intersects $(z_1, 0)$ but not $(0, z_2)$, then $\mathfrak{g}(U)$ is the circle acting by $\theta \cdot (z_1, z_2) = (e^{i\theta} z_1, z_2)$. Similarly if U meets $(z_2, 0)$ but not $(z_1, 0)$. If U meets neither circle, then $\mathfrak{g}(U)$ is the torus. If U meets both circles, then $\mathfrak{g}(U) = \{\mathrm{Id}\}$.

The Solvgeometry Let A be a matrix $A \in SL(2, \mathbb{Z})$, so A can be considered a map $A : T^2 \to T^2$. Let M^3 be its mapping torus. If A is nilpotent, M^3 is a nilmanifold, and it supports an F-structure of rank 1. If A has distinct real eigenvalues, it is a solvmanifold. In this case there is a pure F-structure of rank 2, with exactly two substructures of rank 1, each corresponding to an eigenvalue of A.

A pure F-structure with no polarization

We construct this space is two steps. First let $\mathcal{E}_\theta$ be the flat space $[0,1] \times \mathbb{C} / \sim$, where $\{0\} \times \mathbb{C}$ is identified to $\{1\} \times \mathbb{C}$ via $(0, v) \mapsto (1, e^{2\pi i\theta} v)$. The torus naturally acts on this space, so we get a pure T-structure of nonconstant rank. Any closed subgroup of the torus produces a pure polarized T-structure. Note that $\mathcal{E}_\theta$ is diffeomorphic to $\mathbb{S}^1 \times \mathbb{C}$, though has a different metric structure.

Now consider the space diffeomorphic to $[0,1] \times \mathbb{S}^1 \times \mathbb{C}$, but give each $\theta \times \mathbb{S}^1 \times \mathbb{C}$ the metric structure of $\mathcal{E}_\theta$. Since $\mathcal{E}_0$ is isometric to $\mathcal{E}_1$, we can identify $\{0\} \times \mathbb{S}^1 \times \mathbb{R}^2$ and $\{1\} \times \mathbb{S}^1 \times \mathbb{R}^2$; we will call this space simply $\mathcal{E}$. The torus acts on each slice, but if we follow the action around, the holonomy on the action group is given by the matrix

$$\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix},$$

which has a single eigenvalue which corresponds to the rotation that fixes the the base circle in each $\mathcal{E}_\theta$. Let $\mathcal{G}$ denote the corresponding T-structure. Clearly no polarization exists for $\mathcal{G}$, since the stalk at any point of any base circle is has dimension higher than the dimension of the orbit there. Any substructure must have stalk a subgroup of T with the same holonomy, but the only eigenvalue of the matrix above corresponds to the action that fixed the base circles of the $\mathcal{E}_\theta$. There fore the unique substructure of $\mathcal{G}$ has orbits of rank zero, so is therefore nonpolarized.

Example of a manifold with an F-structure of positive rank but nonzero signature

Consider $\mathbb{CP}^l = \{[z^1, \ldots, z^{l+1}]\}$, on which the torus $T^{l+1} = \{(e^{i\theta_1}, \ldots, e^{i\theta_{l+1}})\}$ acts with kernel $D = \{(e^{i\theta}, \ldots, e^{i\theta})\}$. The fixed points of the action are the $l+1$ points $p_i = [0, \ldots, z^i, \ldots, 0]$. The tangent space at p_i can be identified with $(z^1, \ldots, \hat{z}^i, \ldots, z^{l+1})$, and on this tangent space the torus $S_i = \mathbb{S}^1_1 \times \cdots \times \hat{\mathbb{S}}^1_i \times \cdots \times \mathbb{S}^1_{l+1} \approx T^{l+1}/D$ acts via the standard representation.

Let M_1, M_2 be two copies of $\mathbb{CP}^l$ with a disk D_i around p_i removed. Let $C : \mathbb{C} \to \mathbb{C}$ be complex conjugation. For any i let $F_i = (f_{i,1}, \ldots, f_{i,l+1})$ be a list of numbers that take the value 0 or 1. Identify the disk $D_i \subset M_1$ with $D_2 \subset M_2$ by using the following identification on tangent spaces

$$(z_1, \ldots, \hat{z}^i, \ldots, z^{l+1}) \sim (C^{f_{i,1}}(z_1), \ldots, \hat{z}^i, \ldots, C^{f_{i,l_1}}(z^{l+1})) \tag{11.1}$$

and projecting to the sphere. If each list F_i has $f_{i,i} = 0$ and takes the value 1 an odd number of times, the result is an oriented manifold M^{2l}_F, where $F = (F_1, \ldots, F_{l+1})$. If $l = 2k$ is even, then we compute the signature of M^{4k}_F using the Novikov-Rohlin additivity of the signature, and get $\sigma(M^{4k}_F) = 2\sigma(\mathbb{CP}^{2k}) = 2$.

Chapter 12

F-structures III

March 16, 2010

12.1 Pure polarized collapse

Assume the (possibly noncompact) manifold X admits a pure polarized F-structure. This means $\mathfrak{g}$ is a locally constant sheaf whose orbits all have the same dimension. On such a manifold we can split the metric into two parts $g = g' + h$ where h vanishes on vectors tangent to the orbits and g' vanishes on vectors perpendicular to orbits. Set

$$g_\delta = \delta^2 g' + h. \tag{12.1}$$

Theorem 12.1.1 (Pure Polarized Collapse) *As $\delta \to 0$ the metric g_δ collapses everywhere. Also $\mathrm{dist}_{g_\delta}(p, q)$ decreases with δ, and the sectional curvature is bounded on any compact set.*

Pf

We examine the curvature at the point p by constructing special coordinates near p. Let k denote the dimension of the orbits Let N^{n-k} be any submanifold through p transverse to the orbits. Given coordinates $y^1, \ldots, y^{n-k}$ on N we can extend these coordinate functions to a neighborhood of p by projecting along the orbits. Finally a k-torus acts locally on the orbits themselves; the push-forward of a basis of its Lie algebra is an independent Abelian set of Killing fields parallel to the orbits, and which span the distribution defined by the orbits. The Frobenius theorem says we can integrate these to get the remaining coordinate functions $x^1, \ldots, x^k$ with coordinate fields $\frac{d}{dx^i}$ equal to the original killing fields. We can choose the origin on any orbit to be its point of intersection with N.

The coordinates field $\frac{d}{dy^i} = X_i + V_i$ can be decomposed into a part parallel to the orbits X_i and a part perpendicular to the orbits V_i. Now make the change of coordinates $u^i = \delta x^i$. Then

$$g_\delta \quad = \quad \begin{pmatrix} \left\langle \frac{d}{dx^i}, \frac{d}{dx^j} \right\rangle_{g_1} & \delta \left\langle \frac{d}{dx^i}, \frac{d}{dy^j} \right\rangle_{g_1} \\ \delta \left\langle \frac{d}{dy^i}, \frac{d}{dx^j} \right\rangle_{g_1} & \delta^2 \left\langle X_i, X_j \right\rangle_{g_1} + \left\langle V_i, V_j \right\rangle_{g_1} \end{pmatrix}.$$

As $\delta \to 0$ the metric converges to a warped product metric.

12.2 Polarized collapse

If the polarization is not pure, it means that the various U_α in the atlas are such that the corresponding pure substructures $\mathcal{G}_\alpha$ possibly have different ranks (though the rank is constant on each U_α). We have to modify the metric on each U_α separately, and at the same time push the various U_α away from each other.

Theorem 12.2.1 *If $\mathcal{G}$ is a polarized F-structure on the compact manifold X, then X admits a sequence of metrics g_δ so that*

(1) The manifold (X, g_δ) collapses

(2) $\mathrm{diam}_{g_\delta}(X) < \mathrm{diam}_{g_1}(X)|\log \delta|$

(3) $\mathrm{Vol}_{g_\delta}(X) < \mathrm{Vol}_{g_1}(X)\delta^k|\log \delta|^n$, *some* $k \geq 1$

(4) Sectional curvature $|K|$ is uniformly bounded.

Pf

Let $\{(U_i, \mathcal{G}_i)\}_{i=1}^N$ be an atlas. Let $f_\alpha : U_\alpha \to [1,2]$ be a collection of functions, constant on the orbits of $\mathcal{G}$, so that $f_i = 1$ in a neighborhood of ∂U_i, and so that $\bigcup_i f_i^{-1}(2) = X$. Put

$$\rho_i \quad = \quad \delta^{\log_2 f_i}.$$

We start with the metric $g_0 = \log_2(\delta)\, g$. On U_i we can write

$$g_0 \ = \ g_1' \, + \, h_1,$$

where g_1' is tangent to the orbits of $\mathcal{G}_1$ and h_1 is perpendicular. Then define g_1 by

$$g_1 \quad = \quad \begin{cases} \rho_1^2 g_1' + h_1 & \text{on } U_1 \\ g_0 & \text{on } X \, - \, U_1 \end{cases}$$

Proceed inductively. Once g_{i-1} has been chosen, set $g_{i-1} = g_i' + h_i$ on U_i where g_i' is parallel to the orbits of $\mathcal{G}_i$ and h_i is perpendicular, and put

$$g_i \;=\; \begin{cases} \rho_i^2 g_i' + h_i & \text{on } U_i \\ g_{i-1} & \text{on } X \,-\, U_i \end{cases}$$

Now *(1)*, *(2)*, and *(3)* are obvious, where $k = \min \operatorname{rank} \mathcal{G}_i$.

We check that sectional curvature is bounded. Let $p \in X$; let $l = \dim \mathcal{O}_p$. Let U_i indicate the atlas charts in which p lies. If we work on a normal atlas, we can arrange that $\{\mathcal{G}_j\}$ etc, where the structures have rank $l_1 > \cdots > l_s \geq k$. The metric near p is changed s times, and we will keep track of the changes in curvature as the metric is changed each time.

Let N^{n-l_j} be a submanifold transverse to the orbits of $\mathcal{G}_j$, and choose coordinates $(\underline{x}^1, \ldots, \underline{x}^l, \underline{y}^1, \ldots, \underline{y}^{n-l})$ as before with $p = (0, \ldots, 0)$, where the coordinate fields $\frac{d}{d\underline{x}^i}$ are just the action fields of $\mathcal{G}_\alpha$ and the $\underline{y}^1, \ldots, \underline{y}^{n-l}$ are constant on the orbits of $\mathcal{G}_\alpha$.

First we scale the coordinates

$$x^i \;=\; \log\delta \,\cdot \underline{x}^i \qquad\qquad y^i \;=\; \log\delta \,\cdot \underline{y}^i.$$

In the new coordinates, we still have that $\frac{d\rho_j}{dx^i} = 0$, but also that

$$\begin{aligned}
\frac{d\rho_j}{dy^i} \;&=\; \frac{1}{\log 2}\frac{df_j}{d\underline{y}^i}\frac{1}{f_j}\delta^{\log_2 f_j} \\
\frac{d^2\rho_j}{dy^k dy^i} \;&=\; \frac{1}{\log\delta}\frac{1}{\log 2}\frac{d^2 f_j}{d\underline{y}^k d\underline{y}^i}\frac{1}{f_j}\delta^{\log_2 f_j} \;-\; \frac{1}{\log\delta}\frac{1}{\log 2}\frac{df_j}{d\underline{y}^k}\frac{df_j}{d\underline{y}^i}\frac{1}{f_j^2}\delta^{\log_2 f_j} \\
&\quad + \left(\frac{1}{\log 2}\right)^2 \frac{df_j}{d\underline{y}^k}\frac{df_j}{d\underline{y}^i}\frac{1}{f_j^2}\delta^{\log_2 f_j}.
\end{aligned}$$

Therefore in these coordinates, the functions ρ_j'/ρ_j and ρ_j''/ρ_j are bounded as $\delta \to 0$. Since by the induction assumption the previous metric g_{j-1} has bounded curvature, so does the new metric.

□

12.3 Nonpolarized Collapse

Let $\mathcal{G}$ be an F-structure on the manifold M. We construct what is called a 'slice polarization.'

12.3.1 Pure structure

Let Σ_i be the union of orbits of $\mathcal{G}$ of dimension i. Let Σ_{ϵ_i} denote the set of points of Σ_i a distance of ϵ_i or greater from $\partial\Sigma_i$ (this is a "thickening" of Σ_i). If N is any submanifold let $\nu(N)$ denote the normal bundle. Let S_{ϵ_i,r_i} denote the set $\{v \in \nu(\Sigma_{\epsilon_i})\ s.t.\ \|v\| < r_i\}$, and let Σ_{ϵ_i,r_i} denote the image of S_{ϵ_i,r_i} under the exponential map. If r_i is chosen small enough, the exponential map is a diffeomorphism.

Lemma 12.3.1 *There is an invariant metric g and numbers ϵ_i, r_i so that*

(1) $\bigcup \Sigma_{\epsilon_i,r_i} = M$

(2) If $i < j$, then $\pi_i = \pi_i \circ \pi_j$ on $\Sigma_{\epsilon_i,r_i} \cap \Sigma_{\epsilon_j,r_j}$.

□

Now set $U_i = \Sigma_{\epsilon_i,r_i}$. If $q \in U_i$, then parallel translation from q to $\pi_i(q)$ along a geodesic induces an injection $\mathcal{G}_q \to \mathcal{G}_{\pi_i(q)}$.

Lemma 12.3.2 *There exists an inner product $\langle , \rangle_p$ on g_p, the Lie algebra of stalks $\mathcal{G}_p$, that is invariant under the action of $\mathcal{G}_p$ and under the projections π_i whenever $\pi_i(q)$ is defined.*

□

For $p \in S_{\epsilon_i,r_i}$ let K^i_p be the (not necessarily closed) subgroup of $\mathcal{G}_p$ whose lie algebra is the orthogonal complement of the isotropy group of p. Set $K^i_p = \pi_i^{-1}(K_{\pi_i(p)})$. It follows from the previous lemmas that the assignment $p \to K^i_p$ is invariant under the local action of $\mathcal{G}_p$.

We can now describe the collapsing procedure. Let f_i, ρ_i be as before. Fix q and let $U_{i_1}, \ldots, U_{i_j}$, $i_1 < \cdots < i_j$ be the U_i with $q \in U_i$. Let $Z_{i_1} \subseteq \cdots \subseteq Z_{i_j}$ denote the subspaces of T_qM tangent to the orbits of $K^{i_1}_q, \ldots, K^{i_j}_q$. Let $W_{i_j} \subseteq \cdots \subseteq W_{i_1}$ denote the subspaces $W_{i_1} = \pi_{i_1}^{-1}(\mathcal{O}_{\pi_{i_1}(q)}), \ldots, W_{i_j} = \pi_{i_j}^{-1}(\mathcal{O}_{\pi_{i_j}(q)})$. Note that also $Z_{i_j} \subseteq W_{i_j}$.

Now let g be the invariant metric from Lemma 12.3.1. Set $g_0 = \log^2 \delta \cdot g$, and write a decomposition for g_0

$$g_0 \;=\; g'_1 + h_1 + k_1,$$

corresponding to $Z_{i_1}, Z_{i_1}^{\perp} \cap W_{i_1}, W_{i_1}^{\perp}$. Put

$$g_1 \;=\; \begin{cases} \rho^2 g'_1 + h_1 + \rho^{-2} k_1 & p \in U_1 \\ g_0 & \text{otherwise} \end{cases}$$

Proceed by induction, letting $g_{l-1} = g_l' + h_l + k_l$ be the decomposition according to Z_{i_l}, $Z_{i_l}^{\perp} \cap W_{i_l}$, $W_{i_l}^{\perp}$, and putting

$$g_l = \begin{cases} \rho^2 g_l' + h_l + \rho^{-2} k_l & p \in U_l \\ g_{l-1} & \text{otherwise} \end{cases}$$

First we claim that curvature is bounded as $\delta \to 0$. We establish a coordinate system. Let

$$m_i = \dim \Sigma_i - i = \dim \Sigma_i - \mathrm{rank}_{\mathbb{F}} \Sigma_i,$$

and let $s^1, \ldots, s^{m_{i_1}}$ be coordinates on Σ_{i_1} constant on the orbits. Extend these to U_{i_1} via π_{i_1}. Let $s^{m_{i_1}+1}, \ldots, s^{m_{i_2}}$, be coordinates on U_{i_2}, constant on the orbits. Extend these to $U_{i_1} \cap U_{i_2}$. Proceed in this way, finally getting coordinates $s^1, \ldots, s^{m_{i_j}}$ on $U_{i_1} \cap \cdots \cap U_{i_j}$. Now compliment these coordinates with additional coordinates $t^1, \ldots, t^{n-i_j-m_{i_j}}$ that are constant on the orbits of $K_q^{i_j}$ and so that $s^1, \ldots, s^{i_j}, t^1, \ldots, t^{n-i_j-m_{i_j}}$ is a complete system that is transverse to the orbits of $\mathbb{F}$. Finally let $x^1, \ldots x^{i_j}$ be coordinates so that $\frac{d}{dx^1}, \ldots, \frac{d}{dx^{i_k}}$ are fields generated by the action of $K_q^{i_k}$.

Now we compute the curvature. First consider the change of metric $g_0 \mapsto g_1$. Relabel the coordinates

$$\begin{aligned}
z^1 &= s^1 \\
&\vdots \\
z^{m_{i_1}} &= s^{m_{i_1}} \\
y^1 &= s^{m_{i_1}+1} \\
&\vdots \\
y^{m_{i_j}-m_{i_1}} &= s^{m_{i_j}} \\
y^{m_{i_j}-m_{i_1}+1} &= t^1 \\
&\vdots \\
y^{n-i_j-m_{i_1}} &= t^{n-i_j-m_{i_j}} \\
x^1 & \\
&\vdots \\
x^{i_j}. &
\end{aligned}$$

The orthogonal decomposition of the tangent space given by $Z_{i_1}, Z_{i_1}^{\perp} \cap W_{i_1}, W_{i_1}^{\perp}$ roughly corresponds to the selection of the x, y, z coordinates. Working in the Σ_{i_1} stratum, $x^1, \ldots, x^{i_1}$ are coordinates on the rank i_1 orbits themselves; this roughly corresponds to Z_{i_1}. The subspace $Z_{i_1}^{\perp} \cap W_{i_1}$ is the subspace directly perpendicular to the stratum; this essentially parametrizes the orbits of $\mathbb{F}$ not in Σ_{i_1}, that is, captures the y coordinates, and also captures the remaining x^k. Finally $W_{i_1}^{\perp}$ parametrizes the orbits of Σ_{i_1}; in fact the coordinate functions $\frac{d}{dx^1}, \ldots, \frac{d}{dx^{i_j}}$ project to zero in this space, or else the action of some of the other strata

$\Sigma_{i_2}, \ldots, \Sigma_{i_j}$ would act on the Σ_1 stratum, which is impossible, and also the action fields are tangent to the orbits and $W_{i_1}^{\perp}$ is perpendicular to all obits. Since the y are coordinates on strata and the strata are perpendicular to $W_{i_1}^{\perp}$, we get that $\frac{d}{dy^k}$ is perpendicular to $W_{i_1}^{\perp}$ as well.

Thus we decompose the vectors

$$\begin{aligned} \frac{d}{dx} &= b_x^1 v_{x,1} + b_x^2 v_{x,2} \\ \frac{d}{dy} &= b_y^1 v_{y,1} + b_y^2 v_{y,2} \\ \frac{d}{dz} &= b_z^1 v_{z,1} + b_z^2 v_{z,2} + b_z^3 v_{z,3} \end{aligned}$$

according to the decomposition $Z_{i_1}, Z_{i_1}^{\perp} \cap W_{i_1}, W_{i_1}^{\perp}$. Multiplying the coordinate functions by $\log \delta$, we have again that $|\rho''_{i_1}/\rho_{i_1}|$ and $|\rho'_{i_1}/\rho_{i_1}|$ are bounded. We get the following matrix for g.

$$(\log \delta)^2\, g = \begin{pmatrix} (b_x^1)^2 + (b_x^2)^2 & b_x^1 b_y^1 + b_x^2 b_y^2 & b_x^1 b_z^1 + b_x^2 b_z^2 \\ b_x^1 b_y^1 + b_x^2 b_y^2 & (b_y^1)^2 + (b_y^2)^2 & b_y^1 b_z^1 + b_y^2 b_z^2 \\ b_x^1 b_z^1 + b_x^2 b_z^2 & b_y^1 b_z^1 + b_y^2 b_z^2 & (b_z^1)^2 + (b_z^2)^2 + (b_z^3)^2 \end{pmatrix}$$

therefore

$$g_1 = \begin{pmatrix} \rho^2 (b_x^1)^2 + (b_x^2)^2 & \rho^2 b_x^1 b_y^1 + b_x^2 b_y^2 & \rho^2 b_x^1 b_z^1 + b_x^2 b_z^2 \\ \rho^2 b_x^1 b_y^1 + b_x^2 b_y^2 & \rho^2 (b_y^1)^2 + (b_y^2)^2 & \rho^2 b_y^1 b_z^1 + b_y^2 b_z^2 \\ \rho^2 b_x^1 b_z^1 + b_x^2 b_z^2 & \rho^2 b_y^1 b_z^1 + b_y^2 b_z^2 & \rho^2 (b_z^1)^2 + (b_z^2)^2 + \rho^{-2} (b_z^3)^2 \end{pmatrix}$$

We make the change of coordinates $x \mapsto \rho_{i_1} x$, $z \mapsto \rho_{i_1}^{-1} z$. In the new coordinates the matrix reads

$$g_1 = \begin{pmatrix} (b_x^1)^2 + \rho^{-2} (b_x^2)^2 & \rho b_x^1 b_y^1 + \rho^{-1} b_x^2 b_y^2 & \rho^2 b_x^1 b_z^1 + b_x^2 b_z^2 \\ \rho b_x^1 b_y^1 + \rho^{-1} b_x^2 b_y^2 & \rho^2 (b_y^1)^2 + (b_y^2)^2 & \rho^3 b_y^1 b_z^1 + \rho b_y^2 b_z^2 \\ \rho^2 b_x^1 b_z^1 + b_x^2 b_z^2 & \rho^3 b_y^1 b_z^1 + \rho b_y^2 b_z^2 & \rho^4 (b_z^1)^2 + \rho^2 (b_z^2)^2 + (b_z^3)^2 \end{pmatrix}.$$

We must deal with the $\rho^{-1} b_x^2$ term somehow. As we choose δ differently, the b_x^2 (and the other b_K^i for $K = x, y, z$, $i = 1, 2, 3$) will be different. Let $b_{x,\delta}^2$ denote b_x^2 in he metric g_δ. Since the coordinate fields d/dx^k, $1 \leq k \leq i_1$, are inside of Z_{i_1} to first order, we get

$$\begin{aligned} \lim \frac{b_x^2(q)}{\rho} &= \lim_{\delta \to 0} \frac{b_{x,\delta}^2(q) - b_{x,0}^2(q)}{\rho - 0} \\ &= \lim_{\delta \to 0} \frac{1}{\log \delta} \frac{b_{x,\delta}^2(q) - b_{x,0}^2(q)}{\delta / \log \delta - 0} \\ &= 0 \end{aligned}$$

Letting $\delta \to 0$, the limiting matrix is just

$$g_1 = \begin{pmatrix} (b_x^1)^2 & 0 & 0 \\ 0 & (b_y^2)^2 & 0 \\ 0 & 0 & (b_z^3)^2 \end{pmatrix}.$$

To continue, we now focus attention on U_{i_2}. We readjusts choice of coordinates, so now

$$\begin{aligned}
z^1 &= s^1 \\
&\vdots \\
z^{m_{i_2}} &= s^{m_{i_2}} \\
y^1 &= s^{m_{i_2}+1} \\
&\vdots \\
y^{m_{i_j}-m_{i_2}} &= s^{m_{i_j}} \\
y^{m_{i_j}-m_{i_2}+1} &= t^1 \\
&\vdots \\
y^{n-i_j-m_{i_2}} &= t^{n-i_j-m_{i_j}} \\
x^1 & \\
&\vdots \\
x^{i_j}. &
\end{aligned}$$

One considers the splitting of the tangent space via $Z_{i_2}, Z_{i_2}^{\perp} \cap W_{i_2}, W_{i_2}^{\perp}$, and repeats the computation of the curvature matrix as above.

To see collapse, the idea is that the orbits are almost totally geodesic as the collapsing proceeds. To be specific, let $q \in M$ and let $\mathcal{O}_q$ be its orbit. Choose r so that the exponential map on vectors perpendicular to the orbits is a diffeomorphism on vectors of length $< r$. There is a number c so that $\text{dist}(q, \partial T_{r/2}(\mathcal{O}_q)) > c$. However there is a closed loop that is noncontractible in $T_{r/2}(\mathcal{O}_q)$ and has length $< c'\delta$. For $\delta < c/c'$ this implies there is a noncontractible geodesic in $T_{r/2}(\mathcal{O}_q)$ of length $< c'\delta$; hence the injectivity radius converges to 0 at q.

12.3.2 Nonpure collapse

If the structure is not pure, then we work on a regular atlas $U_1, \ldots, U_A$. Over U_α we have a pure substructure G_α, and we can carry out the procedure above. If we order the atlases so $\mathcal{G}_{1,p} \subset \cdots \subset \mathcal{G}_{A,p}$, then the orbit stratification near p for higher U_α refines that for lower U_α. We must also modify the cutoff functions ρ_i^α to be equal to 1 in some neighborhood of ∂U_α; this way the charts in the atlas are pushed away from each other as well as the strata inside each chart.

Chapter 13

F-structures IV - Collapsing implies existence of an F-structure

March 18, 2010

Theorem 13.0.3 *Given a manifold M^n, there is a decomposition $M^n = K^n \cup H^n$ where H admits an F-structure of positive rank and if $p \in K$, then there is a $c = c(n) < \infty$ such that*

$$\sup_{y \in B_{ci_p}(p)} |\mathrm{Rm}_y|^{1/2} i_p \ \geq \ c^{-1}.$$

We first try to explain the idea behind the proof. The constant $c(n)$ is chosen so that $\sup_{y\in B_{ci_p}(p)} |\,\mathrm{Rm}_y\,|^{1/2} i_p < c^{-1}$ implies $B_{ci_p}(p)$ is *almost flat* in the sense that there exists a quasi-isometry from some flat manifold into some large subset (compared to the injectivity radius) of $B_{ci_p}(p)$. We construct (elementary) F-structures on flat manifolds, which then pass to these almost-flat balls. A technical argument remains on how to "glue" the F-structures together on overlaps. This is achieved by showing that the F-structures' local actions are "almost" the same, in the C^1-sense. Then a stability theorem is used: if a Lie group has two actions that are "close enough" in the C^1-sense, the actions can be perturbed so as to coincide.

Essentially the orbits of the F-structure correspond to the "most collapsed directions."

13.1 F-structures on complete flat manifolds

The "soul theorem" states that a complete manifold M^n of nonnegative curvature is isometric to the total space of the normal bundle of a compact, totally geodesic flat submanifold, called the *soul*, S^k, of M^n.

Let $\pi_1(S^k)$ be the fundamental group (of course $\pi_1(S^k) \approx \pi_1(M^n)$ as $S^k \hookrightarrow M^n$) is a homotopy equivalence). The Bieberbach theorem states that there is an Abelian normal subgroup $A \lhd \pi_1(S^k)$ of finite rank $\leq \lambda(k)$, corresponding to which is a finite cover (of $\leq \lambda(k)$ sheets) of S^k by a torus T^k.

Considered as an Abelian Lie group, T^k acts on itself, although this does not necessarily give rise to a T-structure on the total space of the normal bundle. But Cheeger-Gromov give a method for defining an F-structure can be defined on the normal bundle, based on the existence of short loops. The idea is as follows. An Abelian group $\Gamma = \{\gamma_1, \ldots, \gamma_N\}$ of discrete isometries of the covering space gives rise to a continuous group of commuting isomorphisms of the covering space. Assume Γ is invariant under conjugation with π_1. Let $\triangle \subset \pi_1$ be the discrete Abelian normal subgroup of finite index guaranteed by the Bieberbach theorem, and assume it commutes with Γ. Then Γ gives rise to a torus T_k which acts (possibly noneffectively) on $\mathbb{R}^n / \triangle$. There is an induced action of π_1 on $Aut(T^k)$, so we get an F-structure on $\mathbb{R}^n / pi_1 = M$.

13.2 Locally collapsed regions

Given $y \in M$ and $R > 0$ define the quantity $v(y, R)$ by

$$v(y, R) \triangleq \sup_{x \in B_{Ri_y}(y)} |\mathrm{Rm}_x|^{\frac{1}{2}} i_y.$$

By an *h-quasi-isometry* (for $h \in [1, \infty)$) between Riemannian manifolds U and V will mean a homeomorphism $f : U \to V$ differentiable of degree at least $C^{k,\alpha}$, so that $\frac{1}{h} g_U \leq f^* g_V \leq h g_U$. Of course a 1-quasi-isometry is an isometry.

Lemma 13.2.1 *Given $h > 0$, $k < \infty$, there is a $\delta = \delta(h, k, n)$ and an $R = R(h, k, n)$ so that if $v(y, \delta^{-1}) < \delta$ then there is a flat manifold F with soul S so that*

i) an h-quasi-isometry $f : U \to U_F$ from some subset $y \in U \subset B_{ki_y}(y)$ a neighborhood U_F in F, where also U contains $B_{\frac{1}{4}ki_y}(y)$,

ii) $\mathrm{dist}(f(y), S) \leq R$,

iii) $\mathrm{Diam}(S) \leq R$.

Pf

Assume (*i*) is false. Put $\delta_i = i^{-1}$. By scale invariance we can assume that $i_y = 1$ and $|\,\mathrm{Rm}\,| < 1/i$ on $B_i(y)$, but there is no h-quasi-isometry from any neighborhood of y to any tubular neighborhood $B_{i \cdot i_y}(S)$ of any soul in any flat manifold.

But by Cheeger-Gromov convergence, as $i \to \infty$ the sets $B_i(y)$ converge in the $C^{1,\alpha}$-topology to a complete flat manifold with unit injectivity radius at a point.

Thus for large enough i, there is indeed an h-quasi-isometry from $B_i(y)$ to a subset of this flat manifold.

If (*ii*) or (*iii*) is false, we can repeat the argument. However, in the limiting flat manifold the soul is a finite distance away, so it is clear that we can chose a subset $U_i \subset B_i(y)$ with $y \in U_i$ that maps onto some tubular neighborhood. □

The h-quasi-isometry is actually too weak a notion. It is important that holonomies converge, not just distances. However since the convergence above occurs in the $C^{1,\alpha}$-topology (in particular, in the C^1 topology), holonomies around geodesic loops based at y converge to the respective holonomies in the flat case.

13.3 Joining of locally defined F-structures

In this section I will describe how F-structures are defined locally, and how they are joined together. Pick $h > 0$. Let $p \in M$ and suppose curvature satisfies $|\,\mathrm{Rm}\,| < \delta\, i_p^{-2}$ inside $B_{i_p \delta^{-1}}(p)$. Then there is some flat manifold, Y_p, and an h-quasi-isometry between a some large subset of $U_p \subset B_{i_p \delta^{-1}}(p)$ and a large subset of Y_p.

There is an F-structure on Y_p, however we do not want the entire F-structure. We will consider a loop at p to be a "short loop" if it is a geomdesic lasso and its length is a definite multiple of the injectivity radius. Corresponding to short geodesic loops at p are short almost-geodesic loops in Y_p, which can be homotoped to (nontrivial!) short geodesic loops. If the loops at p have small holonomy, then (by Bieberbach's theorem) the corresponding loops in Y_p have zero holonomy and therefore correspond to geodesic loops in the covering torus, so correspond to an orbit of the F-structure. Let $\gamma_1, \ldots, \gamma_k$ be the loops at p with small holonomy (say, maximal rotation angle $< 1/4$); a simple argument shows this list is nontrivial. Corresponding to these are loops $\tilde\gamma_1, \ldots, \tilde\gamma_k$ in Y_p, corresponding to which is an F-structure of constant rank k. It is this F-structure which passes down to a neighborhood near p.

Now consider two nearby points p, q with overlapping neighborhoods U_p, U_q. Let $\gamma_1^p, \ldots, \gamma_k^p$ and $\gamma_1^q, \ldots, \gamma_l^q$ be the short loops at p, q, respectively, with maximal holonomy angle $< \frac{1}{4}$; these lead to possibly different F-structures on $U_p \cap U_q$, although $U_p \cap U_q$ is saturated for either structure.

We claim is that a third structure exists on a neighborhood of $U_p \cap U_q$, which contains both previous structures. One can "slide" the loops $\gamma_1^p, \ldots, \gamma_k^p$ and $\gamma_1^q, \ldots, \gamma_l^q$ to a point $p' \in U_p \cap U_q$. At p' these loops still have small holonomy and short length, so define an F-structure on a neighborhood of p'.

Now we can replace U_p with $U_p - \overline{U_q}$ and the same with U_p. Repeating this process, we get at least one F-structure defined in a neighborhood of each point, so that if two such structures overlap, then one contains the other.

If $|\operatorname{Rm}|^{1/2} i_x$ is small enough, the orbits of the F-structures will converge in the C^1 sense. A stability theorem (Grove-Karcher (1973)) says that if two Lie groups produce actions that are close enough in the C^1-sense, the actions can be perturbed so as to coincide.

Chapter 14

Singularities of F-structures I - Classification of singularities in dimension 4

March 23, 2010

14.1 Three singularity models

14.1.1 Rong's Structure I

In this case we let $U_{1,k}$ be the solid torus bundle over $\mathbb{S}^1$, constructed as follows. Let $N = D^2 \times \mathbb{S}^1 \times [0,1]$, and identify $\{(r, \theta_1, \theta_2, 0)\}$ with $\{(r, \theta_1, \theta_2, 1)\}$ by the map that fixes r and maps the θ's by the matrix

$$\begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix}. \tag{14.1}$$

Let $\mathcal{F}_{1,k}$ denote the natural F-structure on $U_{1,k}$, which is given on each fiber by the torus action. We easily see that $(U_{1,k}, \mathcal{F}_{1,k})$ is polarizable iff $k = 0$.

14.1.2 Rong's Structure II

Let Y_1, Y_2, and Y_3 be copies of $D^2 \times [0,1] \times \mathbb{S}^1$. The F-structure on Y_1 and Y_3 will be rotation in the $\mathbb{S}^1$ factor, and the F-structure on the Y_2 factor will be the torus acting on the $\mathbb{S}^1$

factor. Join Y_1 to Y_2 by gluing $D^2 \times 1 \times \mathbb{S}^1 \subset Y_1$ to $D^2 \times 0 \times \mathbb{S}^1 \subset Y_2$ via the identity map, and join Y_2 to Y_3 by gluing $D^2 \times 1 \times \mathbb{S}^1 \subset Y_2$ to $D^2 \times 0 \times \mathbb{S}^1 \subset Y_3$, via some automorphism of the torus. This automorphism can be represented by a matrix in $\mathrm{SL}(2, \mathbb{Z})$. If this matrix has two distinct eigenvectors, then there are two distinct polarized substructures. If the matrix in nilpotent, then up to change of basis this matrix is (14.1), and the structure is not polarizable. We call this structure $\mathcal{F}_{2,k}$. It is polarizable iff $k = 0$.

14.1.3 Rong's Structure III

This is simply $D^2 \times \mathbb{S}^1 \times J$ where J is the line or the half-line. The F-structure is the obvious pure T^2 structure. The singular locus is an open or half-open cylinder, and obviously this structure contains a polarized substructure.

14.2 Singularity classification

We consider pure F-structures. Given an F-structure $\mathcal{F}$ let $\mathcal{Z}(\mathcal{F})$ be the singular locus. We'll use $\mathcal{Z}_0$ to indicate a connected component of $\mathcal{Z}(\mathcal{F})$. Let $\mathcal{E}(\mathcal{F})$ be the set of exceptional $\mathbb{S}^1$ orbits that have nontrivial intersection with $\mathcal{Z}(\mathcal{F})$. Put $\mathcal{W}(\mathcal{F}) = \mathcal{Z}(\mathcal{F}) \cup \mathcal{E}(\mathcal{F})$.

14.2.1 Rank 3 structures

We prove that any rank 3 structure has a polarizable substructure. We will work with an invariant metric. Let W_0 be a singular component. We can lift locally to a finite normal cover, on which a 3-torus acts without fixed points. Let $\mathcal{O}_x$ be a singular orbit; we have $\mathrm{Dim}\,\overline{\mathcal{O}}_x$ equals 1 or 2.

If $\mathrm{Dim}\,\overline{\mathcal{O}}_x = 1$, the isotropy group at x is a 2-torus, which acts effectively (and isometrically) on $T_xM^\perp \approx \mathbb{R}^3$. However this is impossible, for $\mathfrak{so}(3)$ has no 2-dimensional abelian subalgebra.

If $\mathrm{Dim}\,\overline{\mathcal{O}}_x = 2$, the isotropy group is a circle, which acts effectively and isometrically on $T_xM^\perp \approx \mathbb{R}^2$. This action has a single fixed point (the origin), so the orbit $\mathcal{O}_x$ is isolated. A disk $D^2 \subset \mathbb{R}^2$ can be identified with a 2-disk in M via the (normal) exponential map. The images of this disk under the action of the various elements of $\mathcal{F}$ give a tubular neighborhood of $\mathcal{O}_x$ the structure of a 2-disk bundle over a 2-torus.

It is possible to find a polarization $\mathcal{P} \subset \mathcal{F}$. Namely, if an open set $U \in \mathcal{O}_x \times D^2$ intersects the exceptional orbit, assign it the group T^2, and if not assign it the group T^3.

14.2.2 Rank 2 structures

If $\mathcal{O}_x$ is an exceptional orbit then $\mathrm{Dim}\,\mathcal{O}_x = 1$. Consider the action of the isotropy group at x on $T_xM^\perp \approx \mathbb{R}^3$. This gives an embedding $\mathbb{S}^1 \hookrightarrow \mathrm{SO}(3)$, so there is a fixed $\mathbb{R}^1$. This means the orbit is not isolated, and the singular locus is $\mathbb{S} \times I$, $\mathbb{R} \times I$, $\mathbb{S} \times \mathbb{R}$, $\mathbb{S} \times \mathbb{R}^+$, the torus, the Klein bottle, or the 2-plane, However we can rule out $\mathbb{R} \times I$ and $\mathbb{R}^2$, because these cannot be foliated by $\mathbb{S}^1$-orbits.

Now we must distinguish between pure and mixed structures. If the F-structure is a pure structure of rank 2, then the finite cylinder is no longer a possibility. If a connected singular locus $\mathcal{Z}_0$ is a torus, then an ϵ-neighborhood $T_\epsilon(\mathcal{Z}_0)$ is a disk bundle over a torus. There is an $\mathbb{S}^1$ action on the singular locus however, which trivializes another direction. This gives $T_\epsilon(\mathcal{Z}_0)$ the structure of a $D^2 \times \mathbb{S}^1$-bundle (solid torus bundle) over $\mathbb{S}^1$. The structure over this neighborhood admits a polarization iff the structure group is solvable.

If the structure is pure but the manifold is not compact, a singular component $\mathcal{Z}_0$ can be an infinite cylinder. In this case the neighborhood $T_\epsilon(\mathcal{Z}_0)$ is a D^2-bundle over the cylinder, and is necessarily trivial. Such a structure is always admits polarization.

If the structure is mixed, then the singular components that are contained in rank 2 neighborhoods can be finite cylinders that abut rank 1 charts. Consider the boundary of $T_\epsilon(Z_0)$ in this case, which can be described as $\mathbb{S}^1 \times D^2 \cup_{f_1} I \times T^2 \cup_{f_2} \mathbb{S}^1 \times D^1$, where the gluing maps are f_1, f_2 are automorphisms of the torus. Up to homotopy this is a lens space $\mathbb{S}^1 \times D^2 \cup_{f_1 f_2^{-1}} \mathbb{S}^1 \times D^2$. The F-structure restricted to this subset is polarizable iff the the $\mathbb{S}^1$ actions induced on the interior by the boundary maps are multiples of each other. This is the case if $f_1 f_2^{-1} \in \mathrm{SL}(2, \mathbb{Z})$ has two distinct eigenvectors, in which case there are two distinct polarized substructures.

Chapter 15

Singularities of F-structures II - Removability of Singularities

April 6, 2010

15.1 Characteristic Forms and Transgressions

Let G be Lie group with algebra $\mathfrak{g}$. Let

$$\mathcal{P} : \mathfrak{g}^{\otimes k} \to \mathbb{R}$$

be a *symmetric invariant polynomial*, which is to say, a map that is

i) Symmetric: $\mathcal{P}(\eta_1, \ldots, \eta_i, \ldots, \eta_j, \ldots, \eta_k) = \mathcal{P}(\eta_1, \ldots, \eta_j, \ldots, \eta_i, \ldots, \eta_k)$

ii) Invariant: $\mathcal{P}(\mathrm{Ad}_g \eta_1, \ldots, \mathrm{Ad}_g \eta_2) = \mathcal{P}(\eta_1, \ldots, \eta_k)$, and

iii) Polynomial: a sum of elementary multilinear maps on $\mathfrak{g}$ of degree k,

where $\eta_1, \ldots, \eta_k \in \mathfrak{g}$ and $g \in G$. The derivative of Ad is ad, so letting $g(t) = \exp(t\theta)$ and putting this into (*ii*) and taking a derivative gives

ii') $\sum \mathcal{P}(\eta_1, \ldots, [\theta, \eta_i], \ldots, \eta_k) = 0.$

Now let M^n be a manifold with structure group G (normally G is $O(n)$, $SO(n)$, or $U(n)$), and let Ω_i be a $\mathfrak{g}$-valued l_i-form for $i \in \{1, \ldots, k\}$. We can define a $\sum l_i$-form

$$\mathcal{P}(\Omega_1, \ldots, \Omega_k)$$

in the obvious way (inserting forms to evaluate the Ω_i to $\mathfrak{g}$, then taking the polynomial). One easily proves that

$$d\,\mathcal{P}(\Omega_i,\ldots,\Omega_k) \;=\; \sum(-1)^{l_1+\cdots+l_{i-1}}\,\mathcal{P}(\Omega_1,\ldots,d\Omega_i,\ldots,\Omega_k),$$

which is the usual rule for wedge products. If θ is a $\mathfrak{g}$-valued 1-form (eg. a connection 1-form), then (*ii*') is

$$ii") \;\sum(-1)^{l_1+\cdots+l_{i-1}}\,\mathcal{P}(\Omega_1,\,\ldots,\,[\theta,\Omega_i],\,\ldots,\,\Omega_k) \;=\; 0.$$

Adding (using the multilinearity), we get

$$d\,\mathcal{P}(\Omega_i,\ldots,\Omega_k) \;=\; \sum(-1)^{l_1+\cdots+l_{i-1}}\,\mathcal{P}(\Omega_1,\ldots,d\Omega_i+[\theta,\Omega_i],\ldots,\Omega_k).$$

If θ is indeed a connection 1-form then $D = d + [\theta,\cdot]$, so we get

$$d\,\mathcal{P}(\Omega_i,\ldots,\Omega_k) \;=\; \sum(-1)^{l_1+\cdots+l_{i-1}}\,\mathcal{P}(\Omega_1,\ldots,D\Omega_i,\ldots,\Omega_k).$$

Therefore $\mathcal{P}(\Omega_1,\ldots,\Omega_k)$ is a closed $(l_1+\cdots+l_k)$-form whenever the Ω_i are covariant-constant (ie. $D\Omega_i = 0$). If $\Omega_i = \Omega = d\theta + \frac{1}{2}[\theta,\theta]$ is the curvature 2-form, then $D\Omega = 0$. Therefore, assigned to each connection is a curvature 2-form and so a deRham class in $H^{2k}(M)$. Given a connection θ let $\mathcal{P}_\theta$ denote the representative $2k$-form.

The question is whether this class is unique. To answer this, let θ_0, θ_1 be two connection 1-forms, and let $\theta_t = t\theta_1 + (1-t)\theta_0$ be the interpolation between them. Corresponding to the connection θ_t is the curvature tensor $\Omega_t = d\theta_t + \frac{1}{2}[\theta_t,\theta_t]$. Since Ω is a form of even degree, we compute

$$\begin{aligned}
\frac{d}{dt}\,\mathcal{P}(\Omega_t,\ldots,\Omega_t) \;&=\; k\,\mathcal{P}(d\Omega_t/dt,\Omega_t,\ldots,\Omega_t)\\
&=\; k\,\mathcal{P}(d(\theta_1-\theta_0)+[\theta_t,\theta_1-\theta_0],\,\Omega_t,\ldots,\Omega_t)\\
&=\; k\,\mathcal{P}(D_t(\theta_1-\theta_0),\,\Omega_t,\ldots,\Omega_t)\\
&=\; kd\,\mathcal{P}(\theta_1-\theta_0,\Omega_t,\ldots,\Omega_t)
\end{aligned}$$

where $D_t\alpha = d\alpha + [\theta_t,\alpha]$. Therefore

$$\mathcal{P}_{\theta_1} - \mathcal{P}_{\theta_0} \;=\; k\,d\int_0^1 \mathcal{P}(\theta_1-\theta_0,\Omega_t,\ldots,\Omega_t)\,dt,$$

and so $\mathcal{P}(\Omega_1,\ldots,\Omega_1)$ and $\mathcal{P}(\Omega_0,\ldots,\Omega_0)$ define the same cohomology class. The $(2k-1)$-form $k\int_0^1 \mathcal{P}(\theta_0-\theta_1,\Omega_t,\ldots,\Omega_t)\,dt$ is often called a *transgression form*, and denoted $\mathcal{TP} = \mathcal{TP}(\theta_1,\theta_0)$. We have

$$\underset{\theta_1}{\mathcal{P}} - \underset{\theta_0}{\mathcal{P}} + d\,\mathcal{TP}(\theta_1,\theta_0) \;=\; 0$$

15.2 Characteristic numbers

If n is even and $G = SO(n)$, let $\mathcal{P}(\eta_1, \ldots, \eta_{n/2})$ be the Pfaffian. If a manifold M^n has structure group $SO(n)$ on its frame bundle then this defines a characteristic class, the Euler class. Put $\mathcal{P}_\chi = \mathcal{P}(\Omega, \ldots, \Omega)$ for the Levi-Civita curvature 2-form Ω; this defines the Euler class. (Of course an Euler class can be defined on any even-dimensional $SO(k)$ principle bundle, but we are only concerned with the frame bundle.) Now let X be a vector field on M with isolated zeros. Replace X with $X/|X|$, so X is defined and C^∞ outside a finite number of singular points. At these singular points the index of $X/|X|$ is defined, and the Euler number of M is the sum of the indices of these singular points. Away from the singularities we have a splitting of the tangent bundle into a parallel and orthogonal distribution. If

$$\theta \;=\; \left(\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right)$$

is the corresponding block decomposition of the the Levi-Civita connection θ, then define a new connection

$$\theta' \;=\; \left(\begin{array}{c|c} A & 0 \\ \hline 0 & D \end{array}\right)$$

Needless to say, $A = 0$, since it is an $\mathfrak{o}(1)$-valued 1-form. Since

$$\Omega' \;=\; \left(\begin{array}{c|c} 0 & 0 \\ \hline 0 & dD + \frac{1}{2}[D, D] \end{array}\right)$$

we have that $\mathcal{P}(\Omega', \ldots, \Omega') = 0$ for any symmetric invariant polynomial $\mathcal{P}$ of degree $\frac{n}{2}$. By the previous section, we have "transgressed" $\mathcal{P}_\chi$ outside the zeros of X:

$$\mathcal{P}_\chi \,+\, d\mathcal{T}\mathcal{P}_\chi \;=\; 0.$$

Letting p_i be the zeros of X and putting $B(i, \epsilon) = B_{p_i}(\epsilon)$, we have

$$\int_{M - \bigcup_i B(i,\epsilon)} \mathcal{P}_\chi \;=\; \sum_i \int_{\partial B(i,\epsilon)} \mathcal{T}\mathcal{P}_\chi.$$

A classical theorem of Weyl gives that

$$\lim_{\epsilon \to 0} \int_{\partial B(i,\epsilon)} \mathcal{T}\mathcal{P}_\chi \;=\; -C\, Ind_{p_i}(X/|X|)$$

where $C = C(n)$ is a constant. Therefore

$$\chi(M) \;=\; \frac{1}{C} \int_M \mathcal{P}_\chi.$$

In dimension 4 it turns out that $C = 8\pi^2$, and $\mathcal{P}_\chi$ is a quadratic functional of the Riemann tensor:

$$\chi\left(M^4\right) \;=\; \frac{1}{8\pi^2} \int_M \frac{1}{24} R^2 \,-\, \frac{1}{2} |\,\overset{\circ}{\mathrm{Ric}}\,|^2 \,+\, |W|^2$$

On the other hand, let $\mathcal{P}_\tau = Tr(\Omega \wedge \Omega)$. It can be proven that $\mathcal{P}_\tau = |W^+|^2 - |W^-|^2$ and

$$\tau \;=\; \frac{1}{3}p_1 \;=\; \frac{1}{12\pi^2}\int_M |W^+|^2 \;-\; |W^-|^2$$

where τ is the signature of the manifold.

15.3 Characteristic numbers of manifolds with boundary

Assume the boundary of M is C^∞. If X is perpendicular to the boundary, it is easy to modify the Weyl formula to get

$$\chi(M) \;=\; \frac{1}{C}\int_M \mathcal{P}_\chi \;+\; \frac{1}{C}\int_{\partial M} \mathcal{T}\mathcal{P}_\chi.$$

On the other hand another term is introduced to the signature formula

$$\tau(M^4) \;=\; \frac{1}{12\pi^2}\int_M |W^+|^2 - |W^-|^2 \;+\; \frac{1}{12\pi^2}\int_{\partial M} \mathcal{T}\mathcal{P}_\tau \;+\; \eta_{\partial M}.$$

The functional $\eta_{\partial M}$ is called the η-invariant. This invariant is defined for any 3-manifold and depends only on the Riemannian structure of ∂M (not how it is embedded as the boundary of M). It is additive over disjoint unions.

Using the Hirzebruch L-polynomial a formula for the signature, in terms of the Riemann tensor, can be obtained for any manifold of dimension $4k$. The corresponding signature formula for $4k$-manifolds with boundary has the boundary corrections coming from both a transgression form and an eta-invariant for the boundary $(4k-1)$-manifolds. See the papers of Atiyah-Patodi-Singer for more information.

15.4 Signatures of the structures $\mathcal{F}_{1,k}$

Let DT^n indicate the solid torus with boundary T^{n-1}. Recall that Rong's non-polarizable structure $\mathcal{F}_{1,k}$ can be considered to be a disk bundle over a 2-torus, or as a solid torus bundle over a circle. As a solid torus bundle, it is

$$\mathcal{F}_{1,k} \;=\; [0,1] \times DT^3/\sim$$

where $\sim$ identifies $\{0\} \times DT^3$ with $\{1\} \times DT^3$ with the matrix

$$\begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix}.$$

To see that is is a disk bundle over a torus, consider the projection on each solid torus that takes DT^3 to its central circle. Since the central circle is mapped to itself isomorphically, this is well-defined globally.

We claim that the signature of this oriented manifold-with-boundary is precisely k. To see this, note that there are two homology classes in $H_2(M, \partial M; \mathbb{Z})$, one of which is carried by the central 2-torus, denoted T, and the other is fiber, denoted D. We claim the intersection form is

$$\begin{pmatrix} \pm k & 1 \\ 1 & 0 \end{pmatrix}.$$

That $D \cdot D = 0$ and $D \cdot T = 1$ are obvious. To see that $T \cdot T = k$ we shall perturb T to another 2-dimensional submanifold T' and show that T and T' intersect transversely in k places, and that the orientations of the intersections are consistent.

Now let S be the meridian circle on the boundary $T^2 \approx \partial(\{1\} \times DT^3)$. This is identified to the circle $S' \subset \partial\partial(\{0\} \times DT^3) \approx T^2$ that wraps around the meridian once and the longitude k times. Let $S(t)$ be a circle in $\{t\} \times DT^3$ with the following property. If $\pi_t : [0,1] \times DT^3 \to DT^3$ is the projection onto the second factor, then the image $\pi_t(S(t))$ is a smooth homotopy from the circle $\pi_0(S(0))$ (that wraps around the boundary $1 - k$ times) and the circle $\pi_1(S(1))$ (that wraps around the boundary $1 - 0$ times), and so that halfway through the homotopy $\pi_t(S(t))$, the circle intersects the boundary circle in precisely k points. Now consider the 2-surface-with-boundary T' that $S(t)$ defines in $[0,1] \times DT^3$. This surface intersects the central cylinder precisely k times. Also, the boundary circle $S(1)$ is identified to the boundary circle $S(0)$ under $\sim$. After identification, we have therefore have $T \cdot T' = \pm k$. It is also clear that T' is smoothly homotopic to T.

15.5 Embedding of $\mathcal{F}_{1,k}$ into a collapsed manifold

Let M be a collapsed manifold with a pure F-structure. All singular irremovable singular orbits are (quotients of) 2-tori, denoted say T, with an exponential tubular neighborhood isomorphic to one of the $\mathcal{F}_{1,k}$. We can prove that there is some $\rho > 0$ so that the injectivity radius for the exponential map off of T is at least ρ.

By the Cheeger-Gromov-Fukaya work on N-structures, there is a critical radius ϵ, so that if this exponential map has injectivity radius $< \epsilon$ then this direction is part of an orbit of a larger N-structure. However, Rong proves that on a definite neighborhood of a singular orbit, the N-structure is in fact just the original F-structure. One way to see this is to note that the singular orbit will remain singular. If there is another collapsed direction, then the N-structure must have a 3-dimensional stalk.

Any singular fiber is therefore 2-dimensional and the isotropy killing fields are therefore 1-dimensional. This implies that a singular fiber is *isolated*. However this is impossible, since the singular fibers of the F-structure are not isolated.

This implies that the normal injectivity radius from the singular locus of the F-structure is at least ϵ.

15.6 Volume bounds

Let Z be a connected component of the singular locus. Then $T_\rho(Z)$ (or a double cover) is diffeomorphic to the structure $\mathcal{F}_{1,k}$. By hypothesis, $T_\rho(Z)$ has very small volume, $|sec| \leq 1$, and boundary diffeomorphic to a nilmanifold. Note that the second fundamental form of the boundary is controlled.

It is possible to extend $T_\rho(Z)$ is a complete manifold of small volume and controlled curvature. Near infinity we can give $T_\rho(Z)$ the structure of an almost-flat manifold crossed with a half-open interval. Using the Atiyah-Patodi-Singer formula

$$\tau(T_\rho(Z)) \;=\; \frac{1}{12\pi^2}\int_M |W^+|^2 - |W^-|^2 \;+\; \frac{1}{12\pi^2}\int_{\partial M} \mathcal{TP}_\tau \;+\; \eta_{\partial M}.$$

we get that η_{N^3}, where $N^3 = \partial T_\rho(Z)$ is very small, where $\mathcal{TP}_\tau = 0$ because the second fundamental form vanishes, and where $\int |W^+|^2 - |W^-|^2$ is controlled by the (very small) volume and bounded sectional curvature. Therefore $|\tau(M^4)|$ is very small and therefore zero, contradicting that $\tau\left(T_\rho(Z)\right) = k$ (unless $k = 0$ and $T_\rho(Z)$ is the trivial disk bundle over the 2-torus).

Chapter 16

Boundary of the space of manifolds of bounded sectional curvature

April 8, 2010

16.1 Boundary consists generically of objects that are locally quotients of manifolds

Let $\mathcal{M}(n, D)$ be the set of Riemannian manifolds with diameter $\leq D$, dimension n, and all sectional curvatures in $[-1, 1]$. By Gromov's precompactness theorem $\mathcal{M}(n, D)$ is precompact in the Gromov-Hausdorff topology, with limiting objects being length spaces. The question is, what is the structure of the length spaces on the boundary?

Let $\tilde{M}_i \in \mathcal{M}(n, D)$ be a sequence that is Cauchy with respect to the Gromov-Hausdorff distance. There is another sequence, M_i, that is ϵ-close in the Lipschitz sense, and has uniformly controlled derivatives of curvature (depending on ϵ). In fact, we have the following theorem

Theorem 16.1.1 *Given any $\epsilon > 0$, if $\tilde{M}^n$ is a Riemannian manifold with $|K| \leq 1$ and Diam$(M) \leq D$, then there is a Riemannian manifold M^n with Lip$(\tilde{M}, M) < \epsilon$ and*

$$|\nabla^k \operatorname{Rm}| \ \leq \ C(n, k, \epsilon). \tag{16.1}$$

<u>*Pf*</u> (somewhat heuristic) Locally lift the metric tensor to a Euclidean ball, smooth via con-

volution with some C_c^∞ function that is C^∞-close to a δ-function, and then pass back down after averaging. This gives a metric that is very close to the original metric, but which has derivatives that depend on the C_c^∞ function that was used. Do this on patches that cover the manifold, being careful to glue the patches together smoothly using a partition of unity argument. The gluing process will not perturb the metric too much, because control can be gained over the multiplicity of the covering. □

This means that given a sequence $\tilde{M}_i$ there is a sequence M_i with $d_{GH}(M_i, \tilde{M}_i) < \epsilon$, but where M_i has uniform C^∞ control on the metric. Then also $d_{GH}(M_\infty, \tilde{M}_\infty) \leq \epsilon$ where $M_\infty = \lim_i M$, $\tilde{M}_\infty = \lim_i \tilde{M}_i$.

Now consider points $p_i \in M_i$ with the p_i converging to some $p \in M_\infty$. Let $B_i \subset T_{p_i} M_i$ be unit balls in the respective tangent spaces, with the pullback metrics, and projections $\pi_i : B_i \to M_i$. Passing to a subsequence if necessary, the metrics on these balls converges in the C^∞ sense (this can be seen using, for instance, Deturck-Kazdan's harmonic coordinate trick). We investigate what happens on the pushdown back to the base space.

Given $p_i \in M_i$ we define G_{p_i}, the *local group at* p_i as follows. A differentiable map $\gamma : U \to B_i$ is *admissible* if $o \in U \subset \frac{1}{2} B_i$ and $\pi_i \circ \gamma = \pi_i$ (in particular, γ is a local isometry). Two admissible maps $\gamma_1 : U_1 \to B_i$ and $\gamma_2 : U_2 \to B_i$ are equivalent if they agree on $U_1 \cap U_2$. An element of G_i is an equivalence class of admissible maps. Any equivalence class is represented by a maximal element. Further, it is possible to define the product of equivalence classes, making G_i into a pseudogroup. The local group G_i partitions B_i, and clearly $\pi_i(B_i) = B_i/G_i$.

Now we can take a limit of the G_i in the following sense. Each element of G_i is represented by a differentiable map $\frac{1}{2} B_i \to B_i$ with uniformly bounded derivatives. Thus we can embed the discrete space G_i into the space $L = C^\infty(\frac{1}{2} B_i, B_i)$. The Arzela-Ascoli theorem indicates that L is compact. Gromov's convergence theorem says that the space of closed subsets of a compact space is compact in the Gromov-Hausdorff topology, so the G_i converge, after possibly passing to another subsequence, so we can assume $G_i \to G$.

It is easily seen that G is a pseudogroup, and it can be proved that G has a differentiable structure. In fact we can prove that G is nilpotent. To do this, we find a point p near $o \in B_\infty$ with trivial isotropy, so that a neighborhood of $\pi_\infty(p)$ is Riemannian. Then the Fukaya-Ruh fibration theorem states that, near p, there is a map $M_i \to M_\infty$ with fibers isomorphic to infranil manifolds, for large enough i. Since G_i embeds into the fundamental group of these infranils, so that k-fold commutators vanish (for k depending only on n, by the proof of Gromov's almost-flat manifold theorem). Since eventually the G_i embed in G and $\bigcap \bigcup G_i$ is dense in G, this implies that k-fold commutators of G also vanish.

We can extend the local group G to a simply connected nilpotent group, and extend the ball B_∞ to an unbounded subset of $\mathbb{R}^n$. We get that locally near p the manifold B_∞ is isomorphic to the quotient of a neighborhood of the origin in $\mathbb{R}^n$ by a simply connected nilpotent Lie group.

16.2 Structure of limits of smooth sequences

More can be said about the structure of limits of smooth manifolds. If the isotropy of G at $p \in B_\infty$ is finite, then a neighborhood of p has a Riemannian structure. If isotropy is not finite, then a neighborhood of p is the quotient of some $\mathbb{R}^k$ by a Lie group whose identity component is a torus in $SO(k)$.

To see this, assume $\xi \in \mathfrak{g}$ satisfies $\xi(p) = p$. Then ξ integrates out to a subgroup of $SO(n)$, which is necessarily compact. Consider the representation of the Lie algebra $\mathbb{C}\,\xi$ on $\mathfrak{g}$ via the adjoint representation.

For some reason, this must be a semisimple representation.

But then there is some eigenvector η, with $\mathrm{ad}_\xi\,\eta = \alpha\eta$. If $\alpha \neq 0$ then the subalgebra $\mathbb{C}\,\xi \oplus \mathbb{C}\,\eta \subset \mathfrak{g}$ is not nilpotent, a contradiction.

16.3 Fibrations over the frame bundle

Let $M_i \to M$ be a smoothed sequence of manifolds converging to the boundary. Over each point $p_i \in M_i$ we have a ball B_i and projections $\pi_i : B_i \to M_i$. On the ball we have pseudogroups G_i with $B_i/G_i \approx \pi_i(B_i)$, and with the G_i converging to a local Lie group G, and the metrics on the balls B_i converging to a metric on B. Then B/G is isometric to a neighborhood of $\lim_i p_i \in M$

Now consider the frame bundles FM_i over the M_i, with a Riemannian metric that comes from the metric on the base space, and a fixed metric on the fibers (which are isomorphic to $O(n)$ or $SO(n)$). Now consider the balls B_i, and consider the pullback bundle FB_i. The pseudogroups G_i act isometrically on B_i, so they therefore act isometrically on FB_i. Furthermore this action is free, since an isometry that fixes a point of FB_i fixes B_i and therefore fixes FB_i as well.

Taking a limit we get that the groups G act freely and isometrically on the limiting bundle FB_∞. This proves that a neighborhood of a point in the limit of the FB_i is a manifold. Furthermore, since we have $FM_i/O(n) \approx M_i$, we also have $FM_\infty/O(n) \approx M_\infty$.

Chapter 17

N-Structures - Definitions

April 13, 2010

17.1 Right fields, left translations

Let G be a Lie group. Now assume $g(t)$ is a 1-parameter family of elements of G. This produces two families of diffeomorphisms, $\lambda_t : G \to G$ and $\rho_t : G \to G$, given respectively by left and right translation: for $h \in G$

$$\begin{aligned} \lambda_t(h) &= L_{g(t)}h = g(t)h \\ \rho_t(h) &= R_{g(t)}h = hg(t). \end{aligned}$$

At a given time t either of these produces a flow field. For simplicity we assume $g(0) = e$, and we compute the flow fields at time 0. Note that $\dot{g}(0) \in T_eG \approx \mathfrak{g}$. Choose a point $h \in G$. By the commutativity of the left and right actions we have

$$\begin{aligned} R_{h^{-1}*}\frac{d}{dt}\Big|_{t=0}\lambda_t(h) &= \frac{d}{dt}\Big|_{t=0}L_{g(t)}R_{h^{-1}}h = \frac{d}{dt}\Big|_{t=0}L_{g(t)}e = \dot{g}(0), \\ L_{h^{-1}*}\frac{d}{dt}\Big|_{t=0}\rho_t(h) &= \frac{d}{dt}\Big|_{t=0}R_{g(t)}L_{h^{-1}}h = \frac{d}{dt}\Big|_{t=0}R_{g(t)}e = \dot{g}(0). \end{aligned}$$

One way to put this is that *right*-invariant vector fields integrate out to 1-parameter families of diffeomorphisms that are in fact *left*-translations, and similarly *left*-invariant fields integrate out to *right*-translations.

Now let X be a left-invariant vector field on G and Y a right-invariant field. We can prove that $[X, Y] = 0$. We have $L_XY = [X, Y]$, so to compute the Lie derivative let $\phi_t : G \to G$ be flow of X. By what we have proven above, $\phi_t = R_{\exp(tX)}$. By assumption

$Y_g = R_{g*}Y$ for some $Y \in \mathfrak{g}$. At a point $g \in G$

$$\begin{aligned} L_X Y &= \lim_{t\to 0} \frac{Y_g - \phi_{t*} Y_{\phi_{-t}(g)}}{t} = \lim_{t\to 0} \frac{Y_g - R_{\exp(tX)*} R_{g \exp(-tX)*} Y}{t} \\ &= \lim_{t\to 0} \frac{Y_g - Y_g}{t} = 0. \end{aligned}$$

This suffices to prove that right-invariant fields are killing fields for left-invariant metrics. If Y is right-invariant and Z, W are left invariant, then $Y(g(Z,W)) = [Y,Z] = [Z,W] = 0$, so

$$(L_Y g)(Z,W) = Y(g(Z,W)) - g([Y,Z],W) - g(Z,[Y,W]) = 0.$$

17.2 (ρ, k)-round metrics

Given positive numbers $\rho \in \mathbb{R}$ and $k \in \mathbb{Z}$, a manifold (M,g) is called (ρ,k)-round at $p \in M$ if there is a subset $V_p \subset M$ containing p, and a normal covering $\phi : \widetilde{V}_p \to V_p$ with deck group Λ_p, such that

i) There is a Lie group H_p with $\Gamma_p \subset H_p$, with finitely many components, and an isometric action on $\widetilde{V}_p$, extending that of Λ_p

ii) H_p is generated by Λ_p and its identity component H_p'

iii) H_p' is nilpotent

iv) V_p contains the ball of radius ρ

v) The injectivity radius at any point of $\widetilde{V}_p$ is greater than ρ

vi) $\sharp(H_p/H_p') = \sharp(\Lambda_p/\Lambda_p \cap H_p') \le k$.

Proposition 17.2.1 *If (M,g) is (ρ,k)-round at p and the injectivity radius at p is $< \rho/k$, then the orbit of any $\tilde{p} \in \pi^{-1}(p)$ has positive dimension.*

Pf

Let $\widetilde{V}_p' = \widetilde{V}_p/\Lambda \cap H_p'$, so therefore $\pi' : \widetilde{V}_p' \to V_p$ is a k-fold covering space. Given $p' \in \pi'^{-1}$ we have $\mathrm{inj}_{p'} \le k\,\mathrm{inj}_p < \rho$. This means $\widetilde{V}_p/\Lambda \cap H_p'$ is a nontrivial quotient, so H_p' is a non-trivial, connected Lie group. □

17.3 Nilpotent killing structures

Let $\mathcal{F}$ be a sheaf of Lie groups over a manifold M. This has an associated sheaf $\mathfrak{g}$ of Lie algebras. An action h of the sheaf $\mathcal{F}$ is a homomorphism from the sheaf $\mathfrak{g}$ into the sheaf of germs of Killing fields.

An *integral curve* γ for the action h is a smooth path such that for every point $p = \gamma(t)$ on the path there is a neighborhood U and a section $X \in h(\mathcal{F}(U))$ with $\dot{\gamma}(t) = X(\gamma(t))$. A set Z is called *invariant* if whenever γ is an integral curve and $\gamma \cap Z \neq \emptyset$, then $\gamma \subset Z$. A unique minimal invariant set containing a point p is called the *orbit* of p, denoted $\mathcal{O}_p$.

If (M, g) is a (ρ, k)-round manifold (M, g), with V_p, N'_p, etc as above, then a sheaf along with its action, $(\mathcal{F}, h)$ defines a *nilpotent killing structure* for (M, g), if for any point $p \in M$ there is an h-invariant neighborhood $U \subset V$ and a normal covering $\widetilde{U} \to U$ with $\widetilde{U} \subset \widetilde{V}$ so that

i) $\pi^*(h)$ is the infinitesimal generator for the action of the group $\pi^*(\widetilde{U})$ whose kernel, K, is discrete, and so that $N'_p = \pi^*(\widetilde{U})/K$, and the action of N'_p is the quotient action,

ii) For all $\widetilde{W} \subset \widetilde{U}$ with $p \in \widetilde{W}$, the structure homomorphism $\pi^*(\widetilde{U}) \to \pi^*(\widetilde{W})$ is an isomorphism,

iii) The neighborhood U and covering $\widetilde{U}$ can be chosen independent of any point in an orbit $\mathcal{O}_p$.

17.4 The Basic Example, and two problems

17.4.1 The basic example

Let N be a simply connected nilpotent Lie group, and suppose $\Lambda \subset N$ is a discrete subgroup. Consider the locally constant sheaf (fiber bundle) with total space $N \times N$. Then Λ acts on this sheaf by $\lambda(n', n) = (\lambda n' \lambda^{-1}, \lambda n)$. Let $\mathcal{F}$ be the quotient sheaf over the quotient nilmanifold $\Lambda \backslash N$. The associated sheaf of lie algebras, $\mathfrak{g}$, is the sheaf of locally defined *right*-invariant vector fields. Let h be the action of $\mathcal{F}$, meaning a sheaf-homomorphism from $\mathfrak{g}$ to the sheaf of (germs of) vector fields on $\Lambda \backslash N$. Any *left*-invariant metric is invariant under this action, and so $\mathfrak{g}$ is a nilpotent Killing structure for any such metric.

17.4.2 First Problem

Fukaya's theorem indicates that the frame bundle of any pointed sequence of manifolds is fibered by (possibly trivial) infranil manifolds. If a manifold is close enough, in the Gromov-Hausdorff sense, to the boundary of the space of Riemannian manifolds (with $|sec| \leq 1$) then this fibration is nontrivial. This fibration is $\mathcal{O}(n)$-equivariant, and passes down to the base manifold (where possibly singular fibers now exist).

In building an N-structure (nilpotent Killing structure) we must decide what the Killing fields are. Restricted to the fibers, these should be the right-invariant fields. On each individual fiber the right-invariant fields can be identified, but it is not clear that this selection is consistent when moving from fiber to fiber. However the identification of the fibers with infranil manifolds is not canonical. This is due to the use of Ruh's theorem. Ruh's theorem identifies an almost flat manifold with an infranil manifold, but in a way that depends on a chosen basepoint.

17.4.3 Second Problem

Fukaya only proves that the fibration of the frame bundle is locally trivial only in bounded regions. If diameters of a collapsing sequence go to infinity, it is possible that different regions of the frame bundle admit fibrations of varying dimension. CFG must find a way to "patch together" different fibrations in different regions.

17.5 Indication of solution to the problems

17.5.1 First Problem

Coming soon.

17.5.2 Second Problem

From the previous subsection, we know that a bounded $O(n)$ invariant set U of the frame bundle admits a fiber bundle structure, with infranil fibers. In addition, the fibration is equivariant (with respect to $O(n)$), and parametrized, meaning the identification of the fibers with a given nilmanifold is determined by solely by the geometry of the fiber, and therefore varies smoothly over U.

In another region U' with $U' \cap U \neq \emptyset$, the fibers on the intersection could be different, and might not even have the same dimension. This is addressed by showing that two such fibrations must be close in the C^1-sense. If this is so then the fibrations can be modified slightly to obtain identical fibrations. It must also be proved that if the two fibrations have different dimensions, then (after a slight modification), the fibers of one are submanifolds of the other. To do this, one must have a way of determining exactly which directions are included in a fibration. But this is achieved by Fukaya's notion of critical radius.

Chapter 18

Einstein Manifolds I

April 15, 2010

18.1 Isoperimetric and Sobolev constants

If Ω is an n-dimensional domain with a Riemannian metric and $\nu > 0$, we define the ν-isoperimetric constant of Ω to be

$$I_\nu(\Omega) \;=\; \inf_{\Omega' \subset\subset \Omega} \frac{\mathrm{Area}(\partial\Omega')}{\mathrm{Vol}(\Omega')^{\frac{\nu-1}{\nu}}}$$

where Area indicates Hausdorff $(n-1)$-measure. If Ω is a closed Riemannian manifold, we take the infimum over domains Ω' with $\mathrm{Vol}\,\Omega' \leq \frac{1}{2}\mathrm{Vol}\,\Omega$; if some such restriction is not made then of course the infimum is zero. Note that if $\nu < n$ then $I_\nu(\Omega) = 0$.

On the other hand we define the ν-Sobolev constant of Ω by

$$S_\nu(\Omega) \;=\; \inf_{f \in C_c^\infty(\Omega)} \frac{\int_\Omega |\nabla f|}{\left(\int_\Omega |f|^{\frac{\nu}{\nu-1}}\right)^{\frac{\nu-1}{\nu}}}.$$

If Ω is a closed Riemannian manifold, we take the infimum over functions with $\mathrm{Vol}(\mathrm{supp}\, f) < \frac{1}{2}\mathrm{Vol}(\Omega)$; if some such restriction is not made then of course the infimum is zero.

Theorem 18.1.1 (Federer-Fleming)

$$I_\nu(\Omega) \;=\; S_\nu(\Omega).$$

Pf

Pf that $S_\nu(\Omega) \le I_\nu(\Omega)$.

With

$$\int |\nabla f| \;\ge\; S_\nu(\Omega) \left(\int f^{\frac{\nu}{\nu-1}} \right)^{\frac{\nu-1}{\nu}},$$

we can let $f \equiv 1$ on Ω', $f \equiv 0$ outside ${\Omega'}^{(\epsilon)}$ (the ϵ-thickening of Ω'), and $f(p) = 1 - \epsilon^{-1}\operatorname{dist}(\Omega', p)$ on ${\Omega'}^{(\epsilon)} - \Omega'$. As $\epsilon \searrow 0$ we have

$$\lim_{\epsilon \searrow 0} \left(\int f^{\frac{\nu}{\nu-1}} \right)^{\frac{\nu-1}{\nu}} \;=\; \operatorname{Vol}(\Omega')^{\frac{\nu-1}{\nu}}$$

$$\lim_{\epsilon \searrow 0} \int |\nabla f| \;=\; \lim_{\epsilon \searrow 0} \frac{\operatorname{Vol}({\Omega'}^{(\epsilon)} - \Omega')}{\epsilon} \;=\; \operatorname{Area}(\partial \Omega').$$

Therefore

$$\operatorname{Area}(\partial\Omega') \;=\; \lim_{\epsilon \searrow 0} \int |\nabla f| \;\ge\; \lim_{\epsilon \searrow 0} S_\nu(\Omega) \left(\int f^{\frac{\nu-1}{\nu}} \right)^{\frac{\nu}{\nu-1}} \;=\; S_\nu(\Omega)\operatorname{Vol}(\Omega')^{\frac{\nu}{\nu-1}}.$$

Pf that $I_\nu(\Omega) \le S_\nu(\Omega)$.

Given a nonnegative C^∞_c function $f : \Omega \to \mathbb{R}$ and given a number t, let $A_t = f^{-1}(t)$ and let $\Omega_t = f^{-1}([t, \infty])$. Locally (near a regular point of f) we can parametrize Ω' by letting f be one coordinate, and putting some coordinates on A_t. We can split the cotangent bundle by letting $df/|df|$ be one covector in an orthonormal coframe. Then if $d\sigma_t$ indicates the wedge product of the remaining vectors, we have Then $dV = \frac{1}{|\nabla f|} df \wedge d\sigma_t$. Therefore

$$\begin{aligned} \int_M |\nabla f|\, dV \;&=\; \int_{min(f)}^{max(f)} \int_{A_t} d\sigma_t\, df \;=\; \int_0^\infty \operatorname{Area}(A_t)\, dt \\ &\ge\; I_\nu(\Omega) \int_0^\infty \operatorname{Vol}(\Omega_t)^{\frac{\nu-1}{\nu}}\, dt \end{aligned}$$

The equality $\int_M |\nabla f|\, dV = \int_0^\infty \operatorname{Area}(A_t)\, dt$ is called the *coarea formula*. Changing the order of integration, *á la* calculus III, gives

$$\begin{aligned} \int f^{\frac{\nu}{\nu-1}} \;&=\; \frac{\nu}{\nu-1} \int_\Omega \int_0^{f(p)} t^{\frac{1}{\nu-1}}\, dt\, dVol(p) \\ &=\; \frac{\nu}{\nu-1} \int_0^\infty \int_{\Omega_t} t^{\frac{1}{\nu-1}}\, dV\, dt \;=\; \frac{\nu}{\nu-1} \int_0^\infty t^{\frac{1}{\nu-1}} \operatorname{Vol}(\Omega_t)\, dt \end{aligned}$$

The result follows from the following lemma.

Lemma 18.1.2 *If $g(t)$ is a nonnegative decreasing function and $s \ge 1$, then*

$$\left(s \int_0^\infty t^{s-1} g(t)\, dt \right)^{\frac{1}{s}} \;\le\; \int_0^\infty g(t)^{\frac{1}{s}}\, dt$$

Pf

We have

$$\frac{d}{dT}\left(s\int_0^T t^{s-1}g(t)\,dt\right)^{\frac{1}{s}} = T^{s-1}g(T)\left(s\int_0^T t^{s-1}g(t)\,dt\right)^{\frac{1}{s}-1}$$
$$\leq T^{s-1}g(T)^{\frac{1}{s}}\left(s\int_0^T t^{s-1}\,dt\right)^{\frac{1}{s}-1} = g(T)^{\frac{1}{s}}.$$

Since $\frac{d}{dT}\int_0^T g(t)^{\frac{1}{s}}\,dt = g(T)^{\frac{1}{s}}$, we have

$$\left(s\int_0^T t^{s-1}g(t)\,dt\right)^{\frac{1}{s}} \leq \int_0^T g(t)^{\frac{1}{s}}\,dt$$

for all T. □

18.2 Sobolev embedding

As long as $1 \leq p < \nu$ we have

$$\left(\int_\Omega |\nabla f|^p\right)^{\frac{1}{k}} \geq |\Omega|^{\frac{1}{p}-1}\int_\Omega |\nabla f|$$
$$\geq S_\nu|\Omega|^{\frac{1}{p}-1}\left(\int_\Omega f^{\frac{\nu}{\nu-1}}\right)^{\frac{\nu-1}{\nu}} = S_\nu|\Omega|^{\frac{1}{p}-\frac{1}{\nu}}\left(\frac{1}{|\Omega|}\int_\Omega f^{\frac{\nu}{\nu-1}}\right)^{\frac{\nu-1}{\nu}}$$
$$\geq S_\nu|\Omega|^{\frac{1}{p}-\frac{1}{\nu}}\left(\frac{1}{|\Omega|}\int_\Omega f^{\frac{p\nu}{\nu-p}}\right)^{\frac{\nu-p}{p\nu}} = S_\nu\left(\int_\Omega f^{\frac{p\nu}{\nu-p}}\right)^{\frac{\nu-p}{p\nu}}.$$

This gives the Sobolev embedding

$$W^{1,p} \hookrightarrow L^{\frac{p\nu}{\nu-p}}.$$

Likewise we have $W^{2,p} \hookrightarrow W^{1,\frac{p\nu}{\nu-p}} \hookrightarrow L^{\frac{p\nu}{\nu-2p}}$ and so forth, giving

$$W^{k,p} \hookrightarrow L^{\frac{p\nu}{\nu-kp}}.$$

Thus we see this holds on any Riemannian manifold, as long as the ν-isoperimetric constant (where $\nu \geq 0$) is controlled.

18.3 The elliptic equation for Einstein metrics

On any Riemannian manifold,

$$\begin{aligned}(\triangle \mathrm{Rm})_{ijkl} &= \mathrm{Rm}_{ijkl,ss} \;=\; \mathrm{Rm}_{ijsl,ks} + \mathrm{Rm}_{ijks,ls} \\ &= \mathrm{Rm}_{ijsl,sk} + \mathrm{Rm}_{ijks,sl} \\ &\quad + \mathrm{Rm}_{skip}\mathrm{Rm}_{pjsl} + \mathrm{Rm}_{skjp}\mathrm{Rm}_{ipsl} + \mathrm{Rm}_{sksp}\mathrm{Rm}_{ijpl} + \mathrm{Rm}_{sklp}\mathrm{Rm}_{ijsp} \\ &\quad + \mathrm{Rm}_{slip}\mathrm{Rm}_{pjks} + \mathrm{Rm}_{sljp}\mathrm{Rm}_{ipks} + \mathrm{Rm}_{slkp}\mathrm{Rm}_{ijps} + \mathrm{Rm}_{slsp}\mathrm{Rm}_{ijkp} \\ &= \mathrm{Ric}_{li,jk} - \mathrm{Ric}_{lj,ik} + \mathrm{Ric}_{kj,il} + \mathrm{Ric}_{kj,jl} + \\ &\quad + \mathrm{Rm}_{skip}\mathrm{Rm}_{pjsl} + \mathrm{Rm}_{skjp}\mathrm{Rm}_{ipsl} + \mathrm{Rm}_{sksp}\mathrm{Rm}_{ijpl} + \mathrm{Rm}_{sklp}\mathrm{Rm}_{ijsp} \\ &\quad + \mathrm{Rm}_{slip}\mathrm{Rm}_{pjks} + \mathrm{Rm}_{sljp}\mathrm{Rm}_{ipks} + \mathrm{Rm}_{slkp}\mathrm{Rm}_{ijps} + \mathrm{Rm}_{slsp}\mathrm{Rm}_{ijkp}\end{aligned}$$

Schematically we can write

$$\triangle \mathrm{Rm} \;=\; \mathrm{Rm} * \mathrm{Rm} + \nabla^2 \mathrm{Ric}\,.$$

In the Einstein case $\mathrm{Ric} = const$, so $\triangle \mathrm{Rm} = \mathrm{Rm} * \mathrm{Rm}$.

If T is a tensor on any Riemannian manifold we have

$$\begin{aligned}|T|\triangle|T| &= \langle T, \triangle T\rangle + |\nabla T|^2 - |\nabla |T||^2 \\ &\geq \langle T, \triangle T\rangle \\ &\geq -|T||\triangle T|.\end{aligned}$$

Putting $f = c(n)|\mathrm{Rm}|$ we therefore have $\triangle f \geq -|f|^2$.

18.4 The L^p theory on Einstein manifolds

For the time being we assume that $|\mathrm{Rm}| \in L^{\frac{n}{2}}$. Later we shall discuss justifications for this assumption. Let ϕ be a C_c^∞ function with $\mathrm{Vol}(\mathrm{supp}\,\phi) \leq \frac{1}{2}\mathrm{Vol}(M)$. The Sobolev inequality gives

$$\begin{aligned}\left(\int \phi^{2\gamma}|\mathrm{Rm}|^{p\gamma}\right)^{\frac{1}{\gamma}} &= S_n^{-2}\int |\nabla(\phi|\mathrm{Rm}|^{\frac{p}{2}})|^2 \\ &\leq 2S_n^{-2}\int |\nabla\phi|^2|\mathrm{Rm}|^p + \frac{p^2}{2}S_n^{-2}\int \phi^2|\mathrm{Rm}|^{p-2}|\nabla|\mathrm{Rm}||^2 \qquad (18.1)\end{aligned}$$

where we use the abbreviation $\gamma = \frac{n}{n-2}$. If f is a positive function it is easy to compute

$$\begin{aligned}(p-1)\int \phi^2 f^{p-2}|\nabla f|^2 &= -2\int \phi f^{p-1}\langle\nabla\phi, \nabla f\rangle - \int \phi^2 f^{p-1}\triangle f \\ &\leq \frac{p-1}{2}\int \phi^2 f^{p-2}|\nabla f|^2 + \frac{2}{p-1}\int |\nabla\phi|^2 f^p - \int \phi^2 f^{p-1}\triangle f \\ \int \phi^2 f^{p-1}|\nabla f|^2 &\leq \left(\frac{2}{p-1}\right)^2\int |\nabla\phi|^2 f^p - \frac{2}{p-1}\int \phi^2 f^{p-1}\triangle f.\end{aligned}$$

With $f = |\operatorname{Rm}|$ and $\triangle|\operatorname{Rm}| \geq -C|\operatorname{Rm}|^2$ we therefore have

$$\int \phi^2 |\operatorname{Rm}|^{p-1} |\nabla f|^2 \quad \leq \quad \left(\frac{2}{p-1}\right)^2 \int |\nabla\phi|^2 |\operatorname{Rm}|^p + \frac{2}{p-1}\int \phi^2 |\operatorname{Rm}|^{p+1}.$$

Putting back into (18.1) we get

$$\frac{S_n^2}{2}\left(\int \phi^{2\gamma}|\operatorname{Rm}|^{p\gamma}\right)^{\frac{1}{\gamma}} \quad \leq \quad \left(1+\left(\frac{2p}{p-1}\right)^2\right)\int |\nabla\phi|^2|\operatorname{Rm}|^p + \frac{2p^2}{p-1}\int \phi^2|\operatorname{Rm}|^{p+1} \tag{18.2}$$

The first step is to put $|\operatorname{Rm}|$ in a slightly higher L^p space.

Lemma 18.4.1 *Assume that Ω is a domain with Sobolev constant $S_n = S_n(\Omega)$, and with*

$$\left(\int_\Omega |\operatorname{Rm}|^{\frac{n}{2}}\right)^{\frac{2}{n}} \leq \frac{1}{4n\gamma} S_n^2.$$

Then if $\phi \in C_c^\infty(\Omega)$ we have

$$\left(\int \phi^{2\gamma} |\operatorname{Rm}|^{\frac{n}{2}\gamma}\right)^{\frac{1}{\gamma}} \quad \leq \quad 4S_n^{-2}\left(1+(2\gamma)^2\right)\int |\nabla\phi|^2 |\operatorname{Rm}|^{\frac{n}{2}}$$

<u>*Pf*</u>

Since $\frac{1}{\gamma}+\frac{2}{n} = 1$ Hölder's inequality gives

$$\frac{1}{2}S_n^2\left(\int \phi^{2\gamma}|\operatorname{Rm}|^{p\gamma}\right)^{\frac{1}{\gamma}} \leq \left(1+\left(\frac{2p}{p-1}\right)^2\right)\int |\nabla\phi|^2|\operatorname{Rm}|^p$$
$$+ \frac{2p^2}{p-1}\left(\int \phi^{2\gamma}|\operatorname{Rm}|^{p\gamma}\right)^{\frac{1}{\gamma}}\left(\int_{\operatorname{supp}\phi}|\operatorname{Rm}|^{\frac{n}{2}}\right)^{\frac{2}{n}}$$
$$\left(\frac{1}{2}S_n^2 - \frac{2p^2}{p-1}\left(\int_{\operatorname{supp}\phi}|\operatorname{Rm}|^{\frac{n}{2}}\right)^{\frac{2}{n}}\right)\left(\int \phi^{2\gamma}|\operatorname{Rm}|^{p\gamma}\right)^{\frac{1}{\gamma}} \leq \left(1+\left(\frac{2p}{p-1}\right)^2\right)\int |\nabla\phi|^2|\operatorname{Rm}|^p.$$

Therefore we require the $L^{n/2}$-norm of $|\operatorname{Rm}|$ to be small compared to p and S_n. If we let $p = n/2$ we get

$$\left(\frac{1}{2}S_n^2 - \frac{n^2}{n-2}\left(\int_{\operatorname{supp}\phi}|\operatorname{Rm}|^{\frac{n}{2}}\right)^{\frac{2}{n}}\right)\left(\int \phi^{2\gamma}|\operatorname{Rm}|^{\frac{n}{2}\gamma}\right)^{\frac{1}{\gamma}} \leq \left(1+\left(\frac{2p}{p-1}\right)^2\right)\int |\nabla\phi|^2|\operatorname{Rm}|^{\frac{n}{2}}.$$

If we require $\left(\int_{\operatorname{supp}\phi}|\operatorname{Rm}|^{\frac{n}{2}}\right)^{\frac{2}{n}} < \frac{1}{4}\frac{1}{n}\frac{1}{\gamma}S_n^2$, then

$$\frac{1}{4}S_n^2\left(\int \phi^{2\gamma}|\operatorname{Rm}|^{\frac{n}{2}\gamma}\right)^{\frac{1}{\gamma}} \leq \left(1+\left(\frac{2n}{n-2}\right)^2\right)\int |\nabla\phi|^2|\operatorname{Rm}|^{\frac{n}{2}}. \tag{18.3}$$

□

Lemma 18.4.2 *There exists a $C = C(n)$ so that if $p \geq \frac{n}{2}$ and*

$$\left(\int_\Omega |\operatorname{Rm}|^{\frac{n}{2}}\right)^{\frac{2}{n}} \;\leq\; \frac{1}{4n\gamma} S_n^2,$$

then

$$\left(\int \phi^{2\gamma} |\operatorname{Rm}|^{p\gamma}\right)^{\frac{1}{\gamma}} \;\leq\; C\, S_n^{-2}\, p^{\frac{n}{2}}\, \sup |\nabla\phi|^2 \int_{\operatorname{supp}\phi} |\operatorname{Rm}|^p$$

<u>*Pf*</u>

We start from (18.2). Since $\frac{1}{\gamma^2} + \frac{2}{n} + \frac{2}{n}\frac{1}{\gamma} = 1$, Hölder's inequality gives

$$\begin{aligned} \int \phi^2 |\operatorname{Rm}|^{p+1} \;&=\; \int \phi^{\frac{2}{\gamma}} |\operatorname{Rm}|^{\frac{p}{\gamma}} |\operatorname{Rm}|^{\frac{2p}{n}} \phi^{\frac{4}{n}} |\operatorname{Rm}| \\ &\leq\; \left(\int \phi^{2\gamma} |\operatorname{Rm}|^{p\gamma}\right)^{\frac{1}{\gamma^2}} \left(\int_{\operatorname{supp}\phi} |\operatorname{Rm}|^p\right)^{\frac{2}{n}} \left(\int \phi^{2\gamma} |\operatorname{Rm}|^{\frac{n}{2}\gamma}\right)^{\frac{2}{n}\frac{1}{\gamma}} \end{aligned}$$

Using (18.2) and the Schwartz inequality,

$$\begin{aligned} &\frac{S_n^2}{2}\left(\int \phi^{2\gamma} |\operatorname{Rm}|^{p\gamma}\right)^{\frac{1}{\gamma}} \\ &\quad\leq\; \left(1 + \left(\frac{2p}{p-1}\right)^2\right) \int |\nabla\phi|^2 |\operatorname{Rm}|^p \;+\; \frac{2p^2}{p-1} \int \phi^2 |\operatorname{Rm}|^{p+1} \\ &\quad\leq\; \left(1 + \left(\frac{2p}{p-1}\right)^2\right) \int |\nabla\phi|^2 |\operatorname{Rm}|^p \;+\; \frac{1}{\gamma}\frac{S_n^2}{2}\left(\int \phi^{2\gamma} |\operatorname{Rm}|^{p\gamma}\right)^{\frac{1}{\gamma}} \\ &\qquad\quad + \frac{2}{n}\left(\frac{S_n^2}{2}\right)^{-\frac{n}{2}\frac{1}{\gamma}} \left(\frac{2p^2}{p-1}\right)^{\frac{n}{2}} \left(\int \phi^{2\gamma} |\operatorname{Rm}|^{\frac{n}{2}\gamma}\right)^{\frac{1}{\gamma}} \int_{\operatorname{supp}\phi} |\operatorname{Rm}|^p. \end{aligned}$$

At this point we note that, if we restrict ourselves to $p \geq \frac{n}{2} \geq$, then $\frac{p}{p-1} \leq 2$. Therefore

$$\begin{aligned} &\frac{S_n^2}{2}\left(\int \phi^{2\gamma} |\operatorname{Rm}|^{p\gamma}\right)^{\frac{1}{\gamma}} \\ &\quad\leq\; \frac{5n}{2}\int |\nabla\phi|^2 |\operatorname{Rm}|^p \;+\; 2^n p^{\frac{n}{2}} \left(\frac{S_n^2}{2}\right)^{-\frac{n}{2}\frac{1}{\gamma}} \left(\int \phi^{2\gamma} |\operatorname{Rm}|^{\frac{n}{2}\gamma}\right)^{\frac{1}{\gamma}} \int_{\operatorname{supp}\phi} |\operatorname{Rm}|^p. \end{aligned}$$

Now using (18.3) we get

$$\begin{aligned} &\frac{S_n^2}{2}\left(\int \phi^{2\gamma} |\operatorname{Rm}|^{p\gamma}\right)^{\frac{1}{\gamma}} \\ &\quad\leq\; \frac{5n}{2}\int |\nabla\phi|^2 |\operatorname{Rm}|^p \;+\; (1 + (2\gamma)^2)\, 2^{n+1}\, p^{\frac{n}{2}} \left(\frac{S_n^2}{2}\right)^{-\frac{n}{2}} \int |\nabla\phi|^2 |\operatorname{Rm}|^{\frac{n}{2}} \int_{\operatorname{supp}\phi} |\operatorname{Rm}|^p \\ &\quad\leq\; \frac{5n}{2}\int |\nabla\phi|^2 |\operatorname{Rm}|^p \;+\; (1 + (2\gamma)^2)\, 2^{n+1} (2n\gamma)^{-\frac{n}{2}}\, p^{\frac{n}{2}} \sup |\nabla\phi|^2 \int_{\operatorname{supp}\phi} |\operatorname{Rm}|^p. \end{aligned}$$

If we let

$$C(n) \;=\; 10n(1+(2\gamma)^2)2^{n+1}(2n\gamma)^{-\frac{n}{2}}$$

then we can write

$$S_n^2\left(\int \phi^{2\gamma}|\,\mathrm{Rm}\,|^{p\gamma}\right)^{\frac{1}{\gamma}} \;\leq\; C\,p^{\frac{n}{2}}\sup|\nabla\phi|^2\int_{\mathrm{supp}\,\phi}|\,\mathrm{Rm}\,|^p.$$

□

Chapter 19

Einstein Manifolds II

April 20, 2010

19.1 Moser Iteration

With S_n being the Sobolev (=isoperimetric) constant, recall the lemma from last time:

Lemma 19.1.1 *There exists a $C = C(n)$ so that if $p \geq \frac{n}{2}$ and*

$$\left(S_n^{-n} \int_\Omega |\operatorname{Rm}|^{\frac{n}{2}} \right)^{\frac{2}{n}} \leq \frac{1}{4n\gamma},$$

then

$$\left(S_n^{-n} \int \phi^{2\gamma} |\operatorname{Rm}|^{p\gamma} \right)^{\frac{1}{p\gamma}} \leq C^{\frac{1}{p}} \, p^{\frac{n}{2p}} \sup |\nabla\phi|^{\frac{2}{p}} \left(S_n^{-n} \int_{\operatorname{supp}\phi} |\operatorname{Rm}|^p \right)^{\frac{1}{p}}$$

We can apply this iteratively to obtain a local C^∞ bound.

Theorem 19.1.2 *There exists a constant $C = C(n)$ so that if*

$$\left(S_n^{-n} \int_\Omega |\operatorname{Rm}|^{\frac{n}{2}} \right)^{\frac{2}{n}} \leq \frac{1}{4n\gamma},$$

then

$$\sup_{B_q(r/2)} |\operatorname{Rm}| \leq C(n) \, r^{-\frac{n}{p}} \left(S_n^{-n} \int_{B_q(r)} |\operatorname{Rm}|^p \right)^{\frac{1}{p}}$$

<u>*Pf*</u>

Given r, let $r_i = \frac{r}{2}\left(1+\frac{1}{2^i}\right)$. Let ϕ_i be a function with $\phi_i \equiv 1$ inside $B_q(r_i)$, $\phi_i \equiv 0$ outside $B_q(r_{i-1})$, and $|\nabla\phi_i| \le 2(r_{i-1}-r_i)^{-1} = r^{-1}2^{i+2}$. Putting

$$\Phi_i \;=\; \left(S_n^{-n}\int_{B_q(r_i)} |\,\mathrm{Rm}\,|^{p\gamma^i}\right)^{\frac{1}{p\gamma^i}},$$

we have from lemma (20.1.1) that

$$\begin{aligned}\Phi_{i+1} \;&\le\; C^{p^{-1}\gamma^{-i}}\left(p\gamma^i\right)^{\frac{n}{2}p^{-1}\gamma^{-i}}\,(4r^{-1}2^i)^{2p^{-1}\gamma^{-i}}\,\Phi_i \\ &=\; (C\,r^{-2}\,p)^{p^{-1}\gamma^{-i}}\,(4\gamma)^{p^{-1}i\gamma^{-i}}\,\Phi_i\end{aligned}$$

Iterating, we get

$$\begin{aligned}\Phi_{i+1} \;&\le\; (C\,r^{-2}\,p)^{\frac{1}{p}\sum_{j=0}^{i}\gamma^{-j}}\,(4\gamma)^{\frac{1}{p}\sum_{j=0}^{i}j\gamma^{-j}}\,\Phi_0 \\ &\le\; (C\,r^{-2}\,p)^{\frac{1}{p}\sum_{j=0}^{\infty}\gamma^{-j}}\,(4\gamma)^{\frac{1}{p}\sum_{j=0}^{\infty}j\gamma^{-j}}\,\Phi_0\end{aligned}$$

We have

$$\begin{aligned}\sum_{j=0}^{\infty}\gamma^{-j} \;&=\; \frac{1}{1-\gamma^{-1}} \;=\; \frac{n}{2} \\ \sum_{j=0}^{\infty}j\gamma^{-j} \;&=\; \frac{\gamma}{(\gamma-1)^2} \;=\; \left(\frac{n}{2}\right)^2\frac{1}{\gamma}.\end{aligned}$$

so that

$$\begin{aligned}\Phi_{i+1} \;&\le\; (C\,r^{-2}\,p)^{\frac{1}{p}\frac{n}{2}}\,(4\gamma)^{\frac{4}{p}\gamma^{-1}n^{-2}}\,\Phi_0 \\ &=\; C(n,p)\,r^{-\frac{n}{p}}\,\Phi_0.\end{aligned}$$

We therefore have

$$\lim_{i\to\infty}\Phi_i \;=\; \lim_{i\to\infty}\left(S_n^{-n}\int_{B_q(r_i)}|\,\mathrm{Rm}\,|^{r\gamma^i}\right)^{\frac{1}{p}\frac{2}{n}} \;=\; \sup_{B_q(r/2)}|\,\mathrm{Rm}\,|.$$

□

We have proven the standard ϵ-regularity lemma:

Theorem 19.1.3 *There exist constants $\epsilon_0 = \epsilon_0(n, S_n)$ and $C = C(n, S_n)$ so that*

$$\int_{B_q(r)}|\,\mathrm{Rm}\,|^{\frac{n}{2}} \;\le\; \epsilon_0$$

implies

$$\sup_{B_q(r/2)} |\,\mathrm{Rm}\,| \;\leq\; C\,r^{-2}\left(\int_{B_q(r)} |\,\mathrm{Rm}\,|^{\frac{n}{2}}\right)^{\frac{2}{n}}.$$

19.2 Kähler geometry

An *almost complex structure* on a manifold is a tensor $J : T_pM \to T_pM$ such that $J^2 = -1$ (namely, $J(J(X)) = -X$ for all $X \in T_pM$); clearly this resembles multiplication by i in $\mathbb{C}^n$. A manifold is a *complex manifold* if is has domains $U_\alpha \subset M$ and maps $\phi_\alpha : U_\alpha \to \mathbb{C}^n$ with transition functions $\phi_{\beta\alpha} = \phi_\beta\phi_\alpha^{-1}$ being *holomorphic*.

A complex manifold automatically carries an almost complex structure: in a coordinate chart $(x^1, y^1, \ldots, x^n, y^n)$ we just define $J\left(\frac{\partial}{\partial x^i}\right) = \frac{\partial}{\partial y^i}$ and $J\left(\frac{\partial}{\partial y^i}\right) = -\frac{\partial}{\partial x^i}$. If the transition functions are holomorphic, then this definition is consistent; the preservation of this definition of J is known as the Cauchy-Riemann condition. But on the other hand, when does the existence of an almost complex structure imply that there are charts with holomorphic transition functions? When this is the case, the almost complex structure is said to be a *complex structure*, or we say that J is *integrable*. The Newlander-Nirenberg theorem provides the answer. Let

$$N(X,Y) \;=\; [JX, JY] \,-\, J[JX, Y] \,-\, J[X, JY] \,-\, [X,Y]$$

be the *Nijenhuis tensor*. On a complex manifold it is automatic that $N \equiv 0$.

Theorem 19.2.1 (Newlander-Nirenberg) *An almost complex manifold is a complex manifold iff $N(X,Y) = 0$ for all smooth vector fields X, Y.*

A metric g on an almost complex manifold is called *Hermitian*, *J-Hermitian*, or *compatible with the almost complex structure* if $g(X,Y) = g(JX, JY)$. In that case we can create the *Kähler form* ω by setting

$$\omega(X,Y) \;=\; g(JX,Y).$$

It is easy to see that the symmetry of g implies the antisymmetry of ω, making it a 2-form. We say that a manifold (M, J, g) is a *Kähler manifold* if J is integrable and if ω is a closed 2-form: $d\omega = 0$.

Note that ω is a *real* 2-form, meaning $\omega(X,Y) \in \mathbb{R}$ whenever X, Y are real sections of $T^{\mathbb{C}}M = TM \otimes \mathbb{C}$. If η is another real form in the same DeRham cohomology class, then of course $\eta - \omega = d\phi$ for some 1-form ϕ. However the so-called $\partial\bar{\partial}$-lemma provides more:

$$\eta \,-\, \omega \;=\; \sqrt{-1}\partial\bar{\partial}\phi$$

for some *function* ϕ. If a fixed Kähler form ω is given, then ϕ is often called the *Kähler potential* for the Kähler form η. Note that for a given potential ϕ the form $\eta = \omega + \sqrt{-1}\partial\bar{\partial}\phi$ is not necessarily the Kähler form of a Riemannian metric, because the associated metric $g_\eta(X,Y) = -\eta(JX,Y)$ (though symmetric) might not be everywhere positive definite.

In a sense, Kähler geometry is the intersection of Riemannian and symplectic geometry. To be more precise, recall that a metric is Kähler if its holonomy is in $U(n)$, Riemannian if its holonomy is in $SO(2n)$, and symplectic if its holonomy is in $Sp(n)$. Note that

$$U(n) \;=\; SO(n) \cap Sp(n),$$

so that a metric with holonomy in both $SO(2n)$ and $Sp(n)$ is Kähler.

19.3 Elliptic systems on canonical manifolds

19.3.1 Extremal Kähler manifolds

On any Kähler manifold, we have

$$\begin{aligned}
\mathrm{Ric}_{i\bar{j},s\bar{s}} &= \mathrm{Ric}_{s\bar{j},i\bar{s}} \\
&= \mathrm{Ric}_{s\bar{j},\bar{s}i} + \mathrm{Rm}_{i\bar{s}s\bar{t}}\mathrm{Ric}_{t\bar{j}} - \mathrm{Rm}_{i\bar{s}t\bar{j}}\mathrm{Ric}_{s\bar{t}} \\
&= \mathrm{Ric}_{s\bar{s},\bar{j}i} + \mathrm{Ric}_{i\bar{t}}\mathrm{Ric}_{t\bar{j}} - \mathrm{Rm}_{i\bar{s}t\bar{j}}\mathrm{Ric}_{s\bar{t}} \\
&= R_{,i\bar{j}} + \mathrm{Ric}_{i\bar{t}}\mathrm{Ric}_{t\bar{j}} - \mathrm{Rm}_{i\bar{s}t\bar{j}}\mathrm{Ric}_{s\bar{t}}.
\end{aligned}$$

$$\begin{aligned}
\mathrm{Ric}_{i\bar{j},\bar{s}s} &= \mathrm{Ric}_{i\bar{s},\bar{j}s} \\
&= \mathrm{Ric}_{i\bar{s},s\bar{j}} - \mathrm{Rm}_{i\bar{s}i\bar{t}}\mathrm{Ric}_{t\bar{s}} + \mathrm{Rm}_{i\bar{s}t\bar{s}}\mathrm{Ric}_{i\bar{t}} \\
&= \mathrm{Ric}_{s\bar{s},i\bar{j}} - \mathrm{Rm}_{i\bar{s}i\bar{t}}\mathrm{Ric}_{t\bar{s}} + \mathrm{Rm}_{i\bar{s}t\bar{s}}\mathrm{Ric}_{i\bar{t}} \\
&= R_{,i\bar{j}} - \mathrm{Rm}_{i\bar{s}i\bar{t}}\mathrm{Ric}_{t\bar{s}} + \mathrm{Rm}_{i\bar{s}t\bar{s}}\mathrm{Ric}_{i\bar{t}}
\end{aligned}$$

Schematically

$$\triangle\,\mathrm{Ric} = \mathrm{Rm} * \mathrm{Ric} + \nabla^2 R.$$

Therefore if the metric is, for instance, CSC Kähler, then we have an elliptic system

$$\triangle\,\mathrm{Rm} \;=\; \mathrm{Rm} * \mathrm{Rm} + \nabla^2\,\mathrm{Ric} \tag{19.1}$$
$$\triangle\,\mathrm{Ric} \;=\; \mathrm{Rm} * \mathrm{Ric}\,. \tag{19.2}$$

It is known that many Kähler manifolds do not admit CSC (much less Kähler-Einstein) Kähler metrics. A generalization of the CSC condition was proposed by Calabi, who proposed minimizing the functional

$$\mathcal{C}(\omega) \;=\; \int R^2 \omega^n$$

over metrics is a fixed class.

If the Kähler metric is extremal in the sense of Calabi, then we do not necessarily have constant scalar curvature, but in fact we have $\triangle X = -Ric(X)$ where $X = R_{,i}$. In this case we have the elliptic system

$$\begin{aligned}\triangle \operatorname{Rm} &= \operatorname{Rm} * \operatorname{Rm} + \nabla^2 \operatorname{Ric} \\ \triangle \operatorname{Ric} &= \operatorname{Rm} * \operatorname{Ric} + \nabla X + \overline{\nabla} X \\ \triangle X &= \operatorname{Ric} * X.\end{aligned}$$

19.3.2 Other cases

There are two other cases of metrics with elliptic systems. The first is the case of metrics with so-called harmonic curvature, namely $\operatorname{Rm}_{ijkl,i} = 0$; this is equivalent to the metric being CSC and $W_{ijkl,i} = 0$.

The other is case of 4-dimensional CSC Bach flat metrics, which includes for isntance the CSC half-conformally flat metrics. There are higher dimensional generalizations of the Bach tensor, but I don't know if making them zero yields an elliptic system.

Chapter 20

Geometric Analysis

April 20, 2010

20.1 Moser Iteration

With S_n being the Sobolev (=isoperimetric) constant, recall the lemma from last time:

Lemma 20.1.1 *There exists a $C = C(n)$ so that if $p \geq \frac{n}{2}$ and*

$$\left(S_n^{-n} \int_\Omega |\operatorname{Rm}|^{\frac{n}{2}} \right)^{\frac{2}{n}} \leq \frac{1}{4n\gamma},$$

then

$$\left(S_n^{-n} \int \phi^{2\gamma} |\operatorname{Rm}|^{p\gamma} \right)^{\frac{1}{p\gamma}} \leq C^{\frac{1}{p}}\, p^{\frac{n}{2p}} \sup |\nabla \phi|^{\frac{2}{p}} \left(S_n^{-n} \int_{\operatorname{supp}\phi} |\operatorname{Rm}|^p \right)^{\frac{1}{p}}$$

We can apply this iteratively to obtain a local C^∞ bound.

Theorem 20.1.2 *There exists a constant $C = C(n)$ so that if*

$$\left(S_n^{-n} \int_\Omega |\operatorname{Rm}|^{\frac{n}{2}} \right)^{\frac{2}{n}} \leq \frac{1}{4n\gamma},$$

then

$$\sup_{B_q(r/2)} |\operatorname{Rm}| \leq C(n)\, r^{-\frac{n}{p}} \left(S_n^{-n} \int_{B_q(r)} |\operatorname{Rm}|^p \right)^{\frac{1}{p}}$$

Pf

Given r, let $r_i = \frac{r}{2}\left(1 + \frac{1}{2^i}\right)$. Let ϕ_i be a function with $\phi_i \equiv 1$ inside $B_q(r_i)$, $\phi_i \equiv 0$ outside $B_q(r_{i-1})$, and $|\nabla \phi_i| \leq 2(r_{i-1} - r_i)^{-1} = r^{-1}2^{i+2}$. Putting

$$\Phi_i \;=\; \left(S_n^{-n} \int_{B_q(r_i)} |\,\mathrm{Rm}\,|^{p\gamma^i} \right)^{\frac{1}{p\gamma^i}},$$

we have from lemma (20.1.1) that

$$\begin{aligned}
\Phi_{i+1} \;&\leq\; C^{p^{-1}\gamma^{-i}} \left(p\gamma^i\right)^{\frac{n}{2}p^{-1}\gamma^{-i}} (4r^{-1}2^i)^{2p^{-1}\gamma^{-i}}\, \Phi_i \\
&=\; (C\, r^{-2}\, p)^{p^{-1}\gamma^{-i}}\, (4\gamma)^{p^{-1}i\gamma^{-i}}\, \Phi_i
\end{aligned}$$

Iterating, we get

$$\begin{aligned}
\Phi_{i+1} \;&\leq\; (C\, r^{-2}\, p)^{\frac{1}{p}\sum_{j=0}^{i}\gamma^{-j}}\, (4\gamma)^{\frac{1}{p}\sum_{j=0}^{i} j\gamma^{-j}}\, \Phi_0 \\
&\leq\; (C\, r^{-2}\, p)^{\frac{1}{p}\sum_{j=0}^{\infty}\gamma^{-j}}\, (4\gamma)^{\frac{1}{p}\sum_{j=0}^{\infty} j\gamma^{-j}}\, \Phi_0
\end{aligned}$$

We have

$$\begin{aligned}
\sum_{j=0}^{\infty} \gamma^{-j} \;&=\; \frac{1}{1-\gamma^{-1}} \;=\; \frac{n}{2} \\
\sum_{j=0}^{\infty} j\gamma^{-j} \;&=\; \frac{\gamma}{(\gamma-1)^2} \;=\; \left(\frac{n}{2}\right)^2 \frac{1}{\gamma}.
\end{aligned}$$

so that

$$\begin{aligned}
\Phi_{i+1} \;&\leq\; (C\, r^{-2}\, p)^{\frac{1}{p}\frac{n}{2}}\, (4\gamma)^{\frac{4}{p}\gamma^{-1}n^{-2}}\, \Phi_0 \\
&=\; C(n,p)\, r^{-\frac{n}{p}}\, \Phi_0.
\end{aligned}$$

We therefore have

$$\lim_{i\to\infty} \Phi_i \;=\; \lim_{i\to\infty} \left(S_n^{-n} \int_{B_q(r_i)} |\,\mathrm{Rm}\,|^{r\gamma^i} \right)^{\frac{1}{p}\frac{2}{n}} \;=\; \sup_{B_q(r/2)} |\,\mathrm{Rm}\,|.$$

□

We have proven the standard ϵ-regularity lemma:

Theorem 20.1.3 *There exist constants $\epsilon_0 = \epsilon_0(n, S_n)$ and $C = C(n, S_n)$ so that*

$$\int_{B_q(r)} |\,\mathrm{Rm}\,|^{\frac{n}{2}} \;\leq\; \epsilon_0$$

implies

$$\sup_{B_q(r/2)} |\,\mathrm{Rm}\,| \;\le\; C\,r^{-2}\left(\int_{B_q(r)} |\,\mathrm{Rm}\,|^{\frac{n}{2}}\right)^{\frac{2}{n}}.$$

20.2 Kähler geometry

An *almost complex structure* on a manifold is a tensor $J : T_pM \to T_pM$ such that $J^2 = -1$ (namely, $J(J(X)) = -X$ for all $X \in T_pM$); clearly this resembles multiplication by i in $\mathbb{C}^n$. A manifold is a *complex manifold* if is has domains $U_\alpha \subset M$ and maps $\phi_\alpha : U_\alpha \to \mathbb{C}^n$ with transition functions $\phi_{\beta\alpha} = \phi_\beta\phi_\alpha^{-1}$ being *holomorphic*.

A complex manifold automatically carries an almost complex structure: in a coordinate chart $(x^1, y^1, \ldots, x^n, y^n)$ we just define $J\left(\frac{\partial}{\partial x^i}\right) = \frac{\partial}{\partial y^i}$ and $J\left(\frac{\partial}{\partial y^i}\right) = -\frac{\partial}{\partial x^i}$. If the transition functions are holomorphic, then this definition is consistent; the preservation of this definition of J is known as the Cauchy-Riemann condition. But on the other hand, when does the existence of an almost complex structure imply that there are charts with holomorphic transition functions? When this is the case, the almost complex structure is said to be a *complex structure*, or we say that J is *integrable*. The Newlander-Nirenberg theorem provides the answer. Let

$$N(X,Y) \;=\; [JX, JY] \,-\, J[JX,Y] \,-\, J[X,JY] \,-\, [X,Y]$$

be the *Nijenhuis tensor*. On a complex manifold it is automatic that $N \equiv 0$.

Theorem 20.2.1 (Newlander-Nirenberg) *An almost complex manifold is a complex manifold iff $N(X,Y) = 0$ for all smooth vector fields X, Y.*

A metric g on an almost complex manifold is called *Hermitian*, *J-Hermitian*, or *compatible with the almost complex structure* if $g(X,Y) = g(JX, JY)$. In that case we can create the *Kähler form* ω by setting

$$\omega(X,Y) \;=\; g(JX,Y).$$

It is easy to see that the symmetry of g implies the antisymmetry of ω, making it a 2-form. We say that a manifold (M, J, g) is a *Kähler manifold* if J is integrable and if ω is a closed 2-form: $d\omega = 0$.

Note that ω is a *real* 2-form, meaning $\omega(X,Y) \in \mathbb{R}$ whenever X, Y are real sections of $T^{\mathbb{C}}M = TM \otimes \mathbb{C}$. If η is another real form in the same DeRham cohomology class, then of course $\eta - \omega = d\phi$ for some 1-form ϕ. However the so-called $\partial\bar{\partial}$-lemma provides more:

$$\eta \,-\, \omega \;=\; \sqrt{-1}\partial\bar{\partial}\phi$$

for some *function* ϕ. If a fixed Kähler form ω is given, then ϕ is often called the *Kähler potential* for the Kähler form η. Note that for a given potential ϕ the form $\eta = \omega + \sqrt{-1}\partial\bar{\partial}\phi$ is not necessarily the Kähler form of a Riemannian metric, because the associated metric $g_\eta(X, Y) = -\eta(JX, Y)$ (though symmetric) might not be everywhere positive definite.

In a sense, Kähler geometry is the intersection of Riemannian and symplectic geometry. To be more precise, recall that a metric is Kähler if its holonomy is in $U(n)$, Riemannian if its holonomy is in $SO(2n)$, and symplectic if its holonomy is in $Sp(n)$. Note that

$$U(n) \;=\; SO(n) \cap Sp(n),$$

so that a metric with holonomy in both $SO(2n)$ and $Sp(n)$ is Kähler.

20.3 Elliptic systems on canonical manifolds

20.3.1 Extremal Kähler manifolds

On any Kähler manifold, we have

$$\begin{aligned}
\mathrm{Ric}_{i\bar{j},s\bar{s}} &= \mathrm{Ric}_{s\bar{j},i\bar{s}} \\
&= \mathrm{Ric}_{s\bar{j},\bar{s}i} + \mathrm{Rm}_{i\bar{s}s\bar{t}}\mathrm{Ric}_{t\bar{j}} - \mathrm{Rm}_{i\bar{s}t\bar{j}}\mathrm{Ric}_{s\bar{t}} \\
&= \mathrm{Ric}_{s\bar{s},\bar{j}i} + \mathrm{Ric}_{i\bar{t}}\mathrm{Ric}_{t\bar{j}} - \mathrm{Rm}_{i\bar{s}t\bar{j}}\mathrm{Ric}_{s\bar{t}} \\
&= R_{,i\bar{j}} + \mathrm{Ric}_{i\bar{t}}\mathrm{Ric}_{t\bar{j}} - \mathrm{Rm}_{i\bar{s}t\bar{j}}\mathrm{Ric}_{s\bar{t}}.
\end{aligned}$$

$$\begin{aligned}
\mathrm{Ric}_{i\bar{j},\bar{s}s} &= \mathrm{Ric}_{i\bar{s},\bar{j}s} \\
&= \mathrm{Ric}_{i\bar{s},s\bar{j}} - \mathrm{Rm}_{i\bar{s}i\bar{t}}\mathrm{Ric}_{t\bar{s}} + \mathrm{Rm}_{i\bar{s}t\bar{s}}\mathrm{Ric}_{i\bar{t}} \\
&= \mathrm{Ric}_{s\bar{s},i\bar{j}} - \mathrm{Rm}_{i\bar{s}i\bar{t}}\mathrm{Ric}_{t\bar{s}} + \mathrm{Rm}_{i\bar{s}t\bar{s}}\mathrm{Ric}_{i\bar{t}} \\
&= R_{,i\bar{j}} - \mathrm{Rm}_{i\bar{s}i\bar{t}}\mathrm{Ric}_{t\bar{s}} + \mathrm{Rm}_{i\bar{s}t\bar{s}}\mathrm{Ric}_{i\bar{t}}
\end{aligned}$$

Schematically

$$\triangle\,\mathrm{Ric} = \mathrm{Rm} * \mathrm{Ric} + \nabla^2 R.$$

Therefore if the metric is, for instance, CSC Kähler, then we have an elliptic system

$$\triangle\,\mathrm{Rm} \;=\; \mathrm{Rm} * \mathrm{Rm} + \nabla^2\,\mathrm{Ric} \qquad (20.1)$$

$$\triangle\,\mathrm{Ric} \;=\; \mathrm{Rm} * \mathrm{Ric}\,. \qquad (20.2)$$

It is known that many Kähler manifolds do not admit CSC (much less Kähler-Einstein) Kähler metrics. A generalization of the CSC condition was proposed by Calabi, who proposed minimizing the functional

$$\mathcal{C}(\omega) \;=\; \int R^2 \omega^n$$

over metrics is a fixed class.

If the Kähler metric is extremal in the sense of Calabi, then we do not necessarily have constant scalar curvature, but in fact we have $\triangle X = -Ric(X)$ where $X = R_{,i}$. In this case we have the elliptic system

$$\begin{aligned}\triangle \,\mathrm{Rm} &= \mathrm{Rm} * \mathrm{Rm} + \nabla^2 \,\mathrm{Ric} \\ \triangle \,\mathrm{Ric} &= \mathrm{Rm} * \mathrm{Ric} + \nabla X + \overline{\nabla} X \\ \triangle X &= \mathrm{Ric} * X.\end{aligned}$$

20.3.2 Other cases

There are two other cases of metrics with elliptic systems. The first is the case of metrics with so-called harmonic curvature, namely $\mathrm{Rm}_{ijkl,i} = 0$; this is equivalent to the metric being CSC and $W_{ijkl,i} = 0$.

The other is case of 4-dimensional CSC Bach flat metrics, which includes for isntance the CSC half-conformally flat metrics. There are higher dimensional generalizations of the Bach tensor, but I don't know if making them zero yields an elliptic system.

Chapter 21

Einstein Manifolds III - Compactness under diameter and volume constraints

April 22, 2010

21.1 Convergence

Assume for the moment that the Sobolev constant is globally controlled. We have

Theorem 21.1.1 *Let M_i be a sequence of n-manifolds with Einstein metrics, with*

i) The Einstein constants λ_i are controlled: $\underline{\lambda} \leq \lambda_i \leq \overline{\lambda}$.

ii) Energy is controlled: $\int_{M_i} |\operatorname{Rm}_i|^{\frac{n}{2}} \leq \Lambda$,

iii) Diameters are bounded from above: $\operatorname{Diam}(M_i) \leq D$

iv) Volumes are bounded from below: $\operatorname{Vol} M_i \geq \nu$

v) The Sobolev constant is controlled: $S_n(M_i) \geq C_S$

Then some subsequence of M_i converges to a length space M_∞. There exists a number $N = N(n, D, \nu, C_S, \Lambda)$ so that away from at most N many point-like singularities, the space M_∞ has the structure of an Einstein manifold, and (i)-(v) continue to hold.

Pf Let r be some small number. We divide M_i into a "good" set $G_i(r)$, and a "bad" set $B_i(r)$ as follows. Given $p \in M_i$, we let $p \in G_i(r)$ if $\int_{B_p(r)} |\,\mathrm{Rm}\,|^{\frac{n}{2}} \leq \epsilon = \epsilon(n, S_n)$, and let $B_i(r) = M_i - G_i(r)$ otherwise. That is,

$$B_i(r) \;=\; \left\{ p \in M_i \;\Big|\; \int_{B_p(r)} |\,\mathrm{Rm}\,|^{\frac{n}{2}} \;>\; \epsilon_0 \right\}.$$

Now cover $B_i(r)$ with balls $\{B_{2r}(p_{i,k})\}_k$ of radius $2r$ in such a way that the half-radius balls $\{B_r(p_{i,k})\}_k$ are disjoint. The procedure for doing this is as follows. Let $p_{i,1} \in M_i$ be any point in $B_i(r)$. If $p_{i,1}, \ldots, p_{i,l}$ are points in $B_i(r)$ with pairwise separation $> 2r$, then let $p_{i,l+1} \in B_i(r)$ be any point that has a distance $> 2r$ away from each $p_{i,1}, \ldots, p_{i,l}$ assuming any such point exists. Clearly this process must end after Λ/ϵ_0 points have been chosen.

Since each of the balls ball $B_(p_{i,k})(2r)$ has volume bounded from above by $C(n)r^n$ (Bishop volume comparison), most of the manifold's volume lies in $G_{i,r}$. Also, $|\,\mathrm{Rm}\,| < \alpha\, r^{-2}$ on $G_{i,r}$, where α can be made as small as desired by adjusting ϵ_0.

Fixing r, a subsequence of the $G_i(r)$ converges in the Gromov-Hausdorff and the $C^{1,\alpha}$-sense to some manifold $G_\infty(r)$. Now let r be smaller, and repeat the process, starting with the subsequence already found. Continuing this with countable many values of r that decline toward 0, a diagonal subsequence will converge to a manifold whose closure is a manifold-with-singularities. The singularities are point-like. Let $M_\infty = \overline{\bigcup_r G_\infty(r)}$ denote the completion of the limit.

The convergence for each choice of r is in the $C^{1,\alpha}$ topology, due to the fact that curvature is bounded. It can be shown that the convergence is actually in the C^∞ sense, using the following argument. Note that curvature is bounded on the interior of each $G_i(r)$. This means that on a ball of definite radius one can pass to a ball in the tangent space with the pullback metric. There we have harmonic coordinates, and the equation

$$\begin{aligned} \triangle(g_{ij}) \;&=\; -2\mathrm{Ric}_{ij} \,+\, Q(g, \partial g) \\ &=\; -2\lambda\, g_{ij} \,+\, Q(g, \partial g) \end{aligned}$$

(following DeTurck-Kazdan, 1981). Since the g_{ij} (and therefore the coefficients on the Laplacian) are controlled in the $C^{1,\alpha}$-sense, Schauder theory gives uniform bounds on $C^{2,\alpha}(g_{ij})$. Bootstrapping this fashion gives uniform $C^{k,\alpha}$ bounds on the functions g_{ij}. □

Note that, ostensibly, curvature grows like $o(r^{-2})$ near the singularities of M_∞.

21.2 The nature of the singularities

With $|\,\mathrm{Rm}\,| = o(r^{-2})$ near the singularities, it is possible to prove the existence of flat tangent cones at the identity. Since we are working in dimension bigger than 2, any such cone must be a standard cone over a quotient of $\mathbb{S}^{n-1}$. However it is *not* possible to prove (in the general case of $|\operatorname{Rm}| = o(r^{-2})$) that a neighborhood of the singular point is homeomorphic to (a neighborhood of) such a cone. In particular the tangent cone need not be unique. However if curvature grows *strictly* slower than r^{-2}, namely $|\operatorname{Rm}| = O(r^{-2+\epsilon})$, or, even better, $|\operatorname{Rm}| < C$, then the Grove-Shiohama theory of critical points allows us to determine that tangent cones are indeed unique.

To improve the growth of $|\operatorname{Rm}|$ we would like to implement the Moser iteration process despite the presence of the point-like singularities on our Einstein manifolds. However this would appear impossible, as we do not know that our elliptic inequality $\triangle u \geq -u^2$ holds weakly across the singularity. Specifically, in the Moser iteration argument, the first stage is the Sobolev inequality, which provides

$$\left(\int \phi^{2\gamma}|\operatorname{Rm}|^{p\gamma}\right)^{\frac{1}{\gamma}} \;\leq\; C\int|\nabla\phi|^2|\operatorname{Rm}|^p \;+\; Cp\int\phi^2\left|\nabla|\operatorname{Rm}|^{\frac{p}{2}}\right|^2.$$

Then it is required that an integration-by-parts be performed on the right-most term, to obtain a Laplacian term. Although the Sobolev inequality is easily seen to hold despite the singularity, integration-by-parts does not. However we have access to the following lemma, first proved in the context of singularities of Yang-Mills instantons.

Theorem 21.2.1 (Sibner's lemma) *Assume 2-sided volume growth bounds, Sobolev constant bounds, and $\triangle u \geq -fu$ where $f \in L^{n/2}(B - \{o\})$ ($B = B_o(r)$ is any ball) and $u \geq 0$. There exists an $\epsilon_0 > 0$ so that if* $\operatorname{supp}\eta \subset B$, *then* $\int_{\operatorname{supp}\eta}|f|^{n/2} < \epsilon_0$ *implies*

$$\int\eta^2|\nabla u^k|^2 \;\leq\; C\int|\nabla\eta|^2|u^k|^2 \tag{21.1}$$

whenever $k > \frac{1}{2}\frac{n}{n-2}$.

□

If the conclusion of Sibner's lemma holds, then clearly the integration-by-parts argument can proceed, and $|\operatorname{Rm}|$ can be bootstrapped into a higher L^p space. Note that $\frac{n}{2} > \frac{n}{n-2}$ when $n > 4$.

In dimension 4 equality holds and Sibner's lemma just fails, so we have to look to the geometry of Einstein manifolds to provide the additional rigidity that can allow improved in regularity. One way this can be found is in an *improved Kato inequality.* The classical (and quite trivial) Kato inequality reads $|\nabla|T||^2 \leq |\nabla T|^2$ for any tensor T. In the case of Einstein manifolds it is possible to improve this inequality:

$$|\nabla\operatorname{Rm}|^2 \;\geq\; (1+\eta)|\nabla|\operatorname{Rm}||^2$$

where $\eta = \eta(n) > 0$ (in fact, $\eta = \frac{1}{3}$ in the 4-dimensional case). The reference is Bando-Kasue-Nakajima, 1989.

To use this information, first note that $|T|\triangle|T| + |\nabla|T||^2 = \langle \triangle T, T\rangle + |\nabla T|^2$ and

$$\begin{aligned}\frac{1}{1-\delta}|T|\triangle|T|^{1-\delta} &= -\delta|\nabla|T||^2 + |T|^{1-\delta}\triangle|T| \\ &= |\nabla T|^2 - (1+\delta)|\nabla|T||^2 + |T|^{-\delta}\langle T, \triangle T\rangle .\end{aligned}$$

Letting $T = \mathrm{Rm}$ we have

$$\begin{aligned}\triangle|T|^{1-\eta} &\geq (1-\eta)|T|^{-1-\eta}\langle T, \triangle T\rangle \\ &\geq -C(n)|T|\cdot|T|^{1-\eta}.\end{aligned}$$

This is again of the form $\triangle u \geq -fu$ where $f \in L^{\frac{n}{2}}$, but now $u \in L^{\frac{1}{1-\eta}\frac{n}{2}}$, and improvement. Therefore Sibner's lemma goes through, even in dimension 4, despite the presence of singularities.

At this point Moser iteration proceeds nearly unchanged and we get $|\,\mathrm{Rm}\,| \in L^\infty_{loc}$ despite the presence of singularities. This implies that the point-like singularities are in fact orbifold points.

If $B_o(r)$ is a ball around a singular point o, one can pass to an orbifold cover. This is a Euclidean ball $\tilde{B}(r)$ around the origin that has a discrete group $\Gamma \subset SO(n)$ and a C^∞ map $\pi : (\tilde{B}(r) - \{pt\}/\Gamma \to B_o(r) - \{o\}$. Let $\tilde{g}_{ij} = \pi^*(g)_{ij}$ be the pullback metric. With $|\widetilde{\mathrm{Rm}}|$ bounded on $\tilde{B}(r)$ we can construct coordinates so that the metric components are $C^{1,1}$ functions. Therefore harmonic coordinates can be constructed (again by the results of Kazdan-DeTurck), in which we have the equation

$$\triangle(g_{ij}) = -2\mathrm{Ric}_{ij} + Q(g, \partial g).$$

Combined with $g_{ij} = \lambda\,\mathrm{Ric}_{ij}$ a bootstrapping argument commences, which gives $g_{ij} \in C^\infty$.

21.3 Various statements of the compactness theorem, and the naturality of the hypotheses

When $|\,\mathrm{Ric}\,|$ is controlled, say $|\,\mathrm{Ric}\,| < \overline{\lambda}$, then the Sobolev constant is controlled in terms of $\overline{\lambda}$, D, and ν; therefore the hypothesis on Sobolev constants is superfluous. The reference for this is Croke, 1980. The defining equation for Einstein metrics is

$$\mathrm{Ric}_{ij} = \lambda g_{ij}$$

Since Ric is scale invariant, if $\lambda \neq 0$ we can scale λ by scaling the metric. The numbers ν and D can also be modified by adjusting the scale, so we can fix just one of the numbers λ, ν, D. In the case that $\lambda \neq 0$ we choose to set $\lambda = \pm 1$, and in the case $\lambda = 0$ we fix $\mathrm{Vol} = 1$. Note that in the case of positive Einstein constant, we have $D \leq \pi/\sqrt{\lambda}$ by Myers' Theorem.

Finally we comment on the energy controls. It is rare that we can control $\int |\operatorname{Rm}|^{\frac{n}{2}}$ on a manifold when $n > 4$. However in the 4-dimensional case, we have the Chern-Gauss-Bonnet integral formula $8\pi^2\chi(M) = \int \frac{1}{24}R^2 - \frac{1}{2}|\overset{\circ}{\operatorname{Ric}}|^2 + |W|^2$. In the Einstein case, this is

$$\chi(M) = \frac{1}{8\pi^2}\int \frac{1}{24}R^2 + |W|^2.$$

It is standard that $|\operatorname{Rm}|^2 = \frac{1}{6}R^2 + 2|\overset{\circ}{\operatorname{Ric}}|^2 + |W|^2$ in dimension 4, but by adjusting the constants slightly,

$$\chi(M) = \int |\operatorname{Rm}|^2.$$

Thus the L^2-norm of the Riemannian curvature is controlled by a topological quantity. We can therefore restate our proposition

Theorem 21.3.1 *Let $\mathcal{M}^n_{-1} = \mathcal{M}(\nu, D, \Lambda)$ be the set of manifolds M such that*

- *M is an n-dimensional Einstein manifold with $\lambda = -1$*
- $\operatorname{Vol}(M) \geq \nu$
- $\operatorname{Diam}(M) \leq D$
- $\int |\operatorname{Rm}|^{\frac{n}{2}} \leq \Lambda$.

Then the conclusions of the Proposition 21.1.1 hold.

Theorem 21.3.2 *Let $\mathcal{M}^n_1 = \mathcal{M}(\nu, \Lambda)$ be the set of manifolds M such that*

- *M is an n-dimensional Einstein manifold with $\lambda = 1$*
- $\operatorname{Vol}(M) \geq \nu$
- $\int |\operatorname{Rm}|^{\frac{n}{2}} \leq \Lambda$.

Then the conclusions of the Proposition 21.1.1 hold.

Theorem 21.3.3 *Let $\mathcal{M}^n_0 = \mathcal{M}(D, \nu, \Lambda)$ be the set of manifolds M such that*

- *M is an n-dimensional Einstein manifold with $\lambda = 0$*
- $\operatorname{Vol}(M) \geq \nu$

- $\mathrm{Diam}(M) \leq D$
- $\int |\operatorname{Rm}|^{\frac{n}{2}} \leq \Lambda$.

Then the conclusions of the Proposition 21.1.1 hold.

Theorem 21.3.4 *Let $\mathcal{M}^4_{-1} = \mathcal{M}(\nu, D, \Lambda)$ be the set of manifolds M such that*

- *M is an 4-dimensional Einstein manifold with $\lambda = -1$*
- $\mathrm{Vol}(M) \geq \nu$
- $\mathrm{Diam}(M) \leq D$
- $\chi(M) \leq \Lambda$.

Then the conclusions of the Proposition 21.1.1 hold.

Theorem 21.3.5 *Let $\mathcal{M}^4_{1} = \mathcal{M}(\nu, \Lambda)$ be the set of manifolds M such that*

- *M is an 4-dimensional Einstein manifold with $\lambda = 1$*
- $\mathrm{Vol}(M) \geq \nu$
- $\chi(M) \leq \Lambda$.

Then the conclusions of the Proposition 21.1.1 hold.

Theorem 21.3.6 *Let $\mathcal{M}^4_{0} = \mathcal{M}(D, \Lambda)$ be the set of manifolds M such that*

- *M is an 4-dimensional Einstein manifold with $\lambda = 0$ and $\mathrm{Vol}(M) = 1$*
- $\mathrm{Diam}(M) \leq D$
- $\chi(M) \leq \Lambda$.

Then the conclusions of the Proposition 21.1.1 hold.

Chapter 22

Einstein Manifolds IV

April 27, 2010

22.1 Sobolev constants on Einstein manifolds

We use the notation VR $B_p(r) \;=\; r^{-n}\,\mathrm{Vol}\;B_p(r)$.

Lemma 22.1.1 *Let M be an Einstein manifold. There are constants $\epsilon_1, C > 0$ that depend only on n so that $\sup_{B_p(r)}|\,\mathrm{Rm}\,| < \epsilon_1\,r^{-2}$ implies $S_n(B_p(r)) > C\cdot(\mathrm{VR}\;B_p(r))^{\frac{1}{n}}$.*

<u>*Pf*</u>

Assume $\pi:\widetilde{M}\to M$ is a k-to-1 covering space where M is a manifold, possibly with boundary. If $\Omega\subset M$ is a domain with boundary $\partial\Omega$, and if $\widetilde{\Omega}=\pi^{-1}(\Omega)$ is its lift, then also $\widetilde{\partial\Omega}=\partial\widetilde{\Omega}$, and we have

$$\begin{aligned}|\partial\Omega| &= k^{-1}|\partial\widetilde{\Omega}|\\ &\geq k^{-1}|\widetilde{\Omega}|^{\frac{n-1}{n}}\\ &= k^{-1}k^{\frac{n-1}{n}}|\Omega|^{\frac{n-1}{n}} \;=\; k^{-\frac{1}{n}}|\Omega|^{\frac{n-1}{n}}\end{aligned}$$

So that $S_n(M)\;\geq\;k^{-\frac{1}{n}}\,S_n(\widetilde{M})$.

The hypotheses of the lemma are scale-invariant so we can assume $|\,\mathrm{Rm}\,|\,\leq\,\epsilon_1$ on $B=B_p(1)$. If the lemma is false, there are examples of such balls B_i with $|\,\mathrm{Rm}_i\,|<\epsilon_i\searrow 0$ $S_n(B_i)\,<\,C_i\,(\mathrm{Vol}\;B_i)^{\frac{1}{n}}$ and $C_i\searrow 0$. If a subsequence of the numbers $\{\mathrm{Vol}\,B_i\}_i$ remains bounded away from 0, the Cheeger lemma implies that injectivity radii are bounded, and we can take a limit. The Sobolev constant is continuous under taking $C^{0,1}$ limits of Riemannian manifolds, and so on this limiting manifold-with-boundary B_∞, we have $S_n(B_\infty)=0$, an impossibility.

Therefore $\mathrm{Vol}\,B_i \searrow 0$, and we are in the collapsing situation. With $|\,\mathrm{Rm}_i\,| \le \epsilon_i$ we know that B_i is almost-flat, and possesses an N-structure. Passing to the frame bundle FB_i over B_i, this N-structure has the following structure. There is a normal cover $\widetilde{FB_i} \to FB_i$ so that $\widetilde{FB_i}$ is the total space of an $\mathbb{R}^k$-bundle over a manifold with controlled injectivity radius.

Let $f_i : B_i \to \mathbb{R}^{\ge 0}$ be a $W_0^{1,1}$ function with $\int |\nabla f_i| \le C_i\,(\mathrm{Vol}\,B_i)^{\frac{1}{n}} \left(\int f_i^{\frac{n}{n-1}}\right)^{\frac{n-1}{n}}$. By normalizing f we can assume $\int f_i^{\frac{n-1}{n}} = 1$. We can lift f_i to an $SO(n)$-invariant function on FB_i. Let $\tilde{f}_i$ be its lift to $\widetilde{FB_i}$. On $\widetilde{FB_i}$ we can restrict to a submanifold $\widetilde{FB}_i^C$ where the fibers are reduced from $\mathbb{R}^k$ to a cube of fixed size. Then $\pi : \widetilde{FB}_i^C \to FB_i$ is (generically) a k-to-1 local homeomorphism, where $k = (\mathrm{Vol}\ SO(n)\ \cdot\ \mathrm{Vol}\ B_i)^{-1}$. Therefore

$$\int |\nabla \widetilde{f}_i| \ \le\ C_i \left(\int \widetilde{f}_i^{\frac{n}{n-1}} \right)^{\frac{n-1}{n}}$$

This is impossible. $\square$

Theorem 22.1.2 *Assume $B_p(r)$ be a ball in a Riemannian manifold, on which the standard ϵ-regularity theorem holds, namely that there are constants $C = C(n) < \infty$ and $\epsilon_0 = \epsilon_0(n) > 0$ so that*

$$S_n(B_p(r)) \int |\,\mathrm{Rm}\,|^{\frac{n}{2}} \ \le\ \epsilon_0$$

implies

$$\sup_{B_p(r/2)} |\,\mathrm{Rm}\,| \ \le\ C\,r^{-2} \left(S_n(B_p(r)) \int_{B_p(r)} |\,\mathrm{Rm}\,|^{\frac{n}{2}} \right)^{\frac{2}{n}}.$$

If we set

$$H = H(p,r) \quad = \quad \sup_{B_q(s)\subset\subset B_p(r)} \frac{1}{\mathrm{VR}\ B_q(s)} \int_{B_q(s)} |\,\mathrm{Rm}\,|^{\frac{n}{2}},$$

then there exist numbers $C_1 = C_1(n) < \infty$ and $\epsilon_1 = \epsilon_1(n) > 0$ so that

$$H < \epsilon \qquad \text{implies} \qquad \sup_{B_p(r/2)} |\,\mathrm{Rm}\,| \ \le\ C_1 r^{-2} H^{\frac{2}{n}}.$$

Pf

We can choose ϵ_1 small enough so that if $|\,\mathrm{Rm}\,| \le C_1 (r/2)^{-2} \epsilon_1^{\frac{2}{n}}$ on each half-radius ball, then the hypotheses of Lemma 22.1.1 are met, and then standard ϵ-regularity gives the conclusion.

If not, then there is a point p_1 so that on the ball $B_{p_1}(r2^{-1})$, the conclusion of the theorem is false. Now consider the half-radius subballs of $B_{p_1}(r2^{-1})$. Let p_2 be a point with $B_{p_2}(r2^{-2}) \subset B_{p_1}(r2^{-1})$ on which the conclusion of the theorem is false. Continuing in this manner we have a sequence of points p_i and balls $B_{p_i}(r2^{-i})$ for which the conclusion is false, meaning $|\,\mathrm{Rm}\,| \geq C_1 2^{2i} r^{-2} \epsilon_1^{\frac{2}{n}}$. However this cannot continue indefinitely, because $|\,\mathrm{Rm}\,|$ is not infinity at any point. Therefore there is a point p_i so that the conclusion is false on $B_{p_i}(r2^{-i})$ but so that the conclusion is true on every subball of half-radius. Now we are in the situation of the previous paragraph, and we have a contradiction. □

22.2 Notations and notions of collapsing

We define $\mathrm{VR}\, B_p(r) = r^{-n} \,\mathrm{Vol}\, B_p(r)$. Define $r_{|R|}(p)$, called the *local curvature radius*, by

$$r_{|R|}(p) \;=\; \sup\, \left\{ r > 0 \;\middle|\; r^{-2}|\,\mathrm{Rm}\,| \;\leq\; 1 \text{ on } B_p(r) \right\}.$$

This is the largest number μ so that scaling the metric by μ^2 produces a ball $B_p(1)$ with $|\,\mathrm{Rm}\,| \leq 1$. Note that $r_{|R|}(p) = \infty$ iff the manifold is flat. A slightly different notion is the *s-local curvature radius*,

$$r^s_{|R|}(p) \;=\; \sup\, \left\{ 0 < r < s \;\middle|\; r^{-2}|\,\mathrm{Rm}\,| \;\leq\; 1 \text{ on } B_p(r) \right\}.$$

This is used in case where we intentionally want to restrict the scale. We can also define the *local energy radius* $\rho(p)$ by

$$\rho(p) \;=\; \sup\, \left\{ r > 0 \;\middle|\; \frac{1}{\mathrm{VR}\, B_p(r)} \int_{B_p(r)} |\,\mathrm{Rm}\,|^{\frac{n}{2}} \;<\; \epsilon_0 \right\}$$

and the *s-local energy radius* $\rho^s(p)$ by

$$\rho^s(p) \;=\; \sup\, \left\{ 0 < r < s \;\middle|\; \frac{1}{\mathrm{VR}\, B_p(r)} \int_{B_p(r)} |\,\mathrm{Rm}\,|^{\frac{n}{2}} \;<\; \epsilon_0 \right\}.$$

This is the largest ball (of radius $\leq s$) on which the standard ϵ-regularity theorems are guaranteed to hold.

A set $E \subset M^n$ is said to be *v-collapsed on the scale r* if

$$p \in E,\; s \leq r \quad\Longrightarrow\quad \mathrm{VR}\, B_p(s) \;\leq\; v.$$

If no scale is mentioned, it is understood that the scale is 1. The set E is said to be *v-collapsed with locally bounded curvature* if

$$p \in E,\; s \leq r_{|R|}(p) \quad\Longrightarrow\quad \mathrm{VR}\, B_p(s) \;\leq\; v$$

and E is said to be (v,σ)-*collapsed with locally bounded curvature* if

$$p \in E,\ s \le r^{\sigma}_{|R|}(p) \quad \Longrightarrow \quad \mathrm{VR}\, B_p(s) \ \le \ v.$$

22.3 Statement of the Cheeger-Tian results

Throughout we assume M^4 is a compact Einstein manifold, normalized to have Einstein constant $\lambda \in \{0, 3, -3\}$. In the case $\lambda = 0$ we normalize so $\mathrm{Vol}(M^4) = 1$.

Theorem 22.3.1 (ϵ-regularity) *There exists numbers ϵ, c so that when $p \in M$ and $r < 1$,*

$$\int_{B_r(p)} |\,\mathrm{Rm}\,|^2 \ \le \ \epsilon,$$

implies

$$\sup_{B_{r/2}(p)} |\,\mathrm{Rm}\,| \ \le \ c\, r^{-2}.$$

Theorem 22.3.2 (Collapse implies concentration of curvature) *There are constants $v > 0$, $\beta < \infty$, $c < \infty$ so that*

$$s^{-4}\ \mathrm{Vol}\ B_s(p) \ \le \ v$$

for all $p \in M$, $s < 1$ implies there are points $p_1, \dots, p_N \in M$ with

$$N \ \le \ \beta \int_M |\,\mathrm{Rm}\,|^2$$

such that

$$\int_{M - \bigcup B_s(p_i)} |\,\mathrm{Rm}\,|^2 \ \le \ c \sum_{i=1}^{N} s^{-4}\ \mathrm{Vol}\ B_s(p_i).$$

Theorem 22.3.3 (Noncollapsing) *There exists a constant $w > 0$ so that $|\lambda| = 3$ implies there is some point p with*

$$\mathrm{Vol}\ B_1(p) \ \ge \ w \cdot \frac{\mathrm{Vol}\ M}{\int_M |\,\mathrm{Rm}\,|^2}$$

Lemma 22.3.4 (The 'Key Estimate') *There is a $c < \infty$, $\delta > 0$, $t > 0$ so that, whenever $E \subset M$ is a bounded open subset, $T_r(E)$ is t-collapsed on the scale r, and*

$$\int_{B_r(p)} |\,\mathrm{Rm}\,|^2 \ \le \ \delta$$

for all $p \in E$, *then*

$$\int_E |\,\mathrm{Rm}\,|^2 \;\leq\; c\,r^{-4}\ \mathrm{Vol}\,(A_r(E))\,.$$

We use $T_s(E)$ indicates the s-tube around E (the set of points of distance $< s$ from E), and we use $A_r(E)$ to denote the "annulus" of radius r around E: $A_r(E) = T_r(E) - \overline{E}$. A set E is said to be t-collapsed on the scale a if $p \in E$ implies $a^{-4}\,\mathrm{Vol}\,B_a(p) \leq t$. We say E is t-collapsed if it is t-collapsed on the scale 1.

Chapter 23

Einstein Manifolds V – Good Chopping

April 29, 2010

23.1 F-structures and the Chern-Gauss-Bonnet theorem

We begin with a useful lemma that bounds the a-local curvature radius in terms of the a-local energy radius.

Lemma 23.1.1 *After possibly rechoosing ϵ_0 in the statement of the standard ϵ-regularity theorem, we have $r^a_{|R|}(p) \ \geq \ \frac{1}{2}\rho^a(p)$.*

<u>*Pf*</u>

Since

$$\frac{1}{\mathrm{VR}\ B_p(\rho^a(p))}\int_{B_p(\rho^a(p))} |\,\mathrm{Rm}\,|^{\frac{n}{2}} \ \leq \ \epsilon_0,$$

the standard ϵ-regularity theorem states that $|\,\mathrm{Rm}\,| < C(\rho^a(p))^{-2}\epsilon_0^{\frac{2}{n}}$ on $B_p(\frac{1}{2}\rho^a(p))$. By choosing ϵ_0 smaller, we have $|\,\mathrm{Rm}\,| < 4r^{-2} = (\rho^a(p)/2)^{-2}$ on $B_p(\frac{1}{2}\rho^a(p))$. By the definition of $r^a_{|R|}(p)$, we have $r^a_{|R|}(p) < \frac{1}{2}\rho^a(p)$. □

Next we prove a lemma saying that collapsing on a fixed scale a plus small energy imply collapsing on the scale of the a-local curvature scale.

Lemma 23.1.2 *If a set K is t-collapsed on the scale a and $\int_{B_p(a)} |\mathrm{Rm}|^{\frac{n}{2}} < \delta$, then it is $\tilde{t} = \tilde{t}(n, \delta, a)$-collapsed with locally bounded curvature. More precisely,*

$$\mathrm{VR}\ B_p(r^a_{|R|}(p)) \ \leq\ \max\left\{2^n t,\ 2^n \delta \epsilon_0^{-1} \theta(a)\right\}$$

where $\theta(a) = a^{-n} \frac{\mathrm{Vol}^{-1} B(a)}{\mathrm{Vol}^0 B(a)}$.

<u>*Pf*</u>

If $\rho^a(p) = a$ then the previous theorem gives $r^a_{|R|}(p) \in [a/2, a]$, so

$$\mathrm{Vol}\ B_p(r^a_{|R|}(p)) \quad \leq \quad \mathrm{Vol}\ B_p(a) \ \leq\ a^n\, t \ \leq\ 2^n \left(r^a_{|R|}(p)\right)^n\ t,$$

so the conclusion holds. If $\rho^a(p) < a$ then

$$\begin{aligned}
\epsilon_0 \quad &= \quad \frac{1}{\mathrm{VR}\ B_p(\rho^a(p))} \int_{B_p(\rho^a(p))} |\mathrm{Rm}|^{\frac{n}{2}} \\
\mathrm{VR}\ B_p(\rho^a(p)) \quad &\leq \quad \epsilon_0^{-1}\delta.
\end{aligned}$$

Relative volume comparison gives

$$\begin{aligned}
\mathrm{Vol}\ B_p(r^a_{|R|}(p)) \quad &\leq \quad \mathrm{Vol}\ B_p(\rho^a(p)/2)\, \frac{\mathrm{Vol}^{-1} B(r^a_{|R|}(p))}{\mathrm{Vol}^{-1} B(\rho^a(p)/2)} \\
\mathrm{VR}\ B_p(r^a_{|R|}(p)) \quad &\leq \quad \mathrm{VR}\ B_p(\rho^a(p)/2)\, \frac{(r^a_{|R|}(p))^{-n}\, \mathrm{Vol}^{-1} B(r^a_{|R|}(p))}{(\rho^a(p)/2)^{-n}\, \mathrm{Vol}^{-1} B(\rho^a(p)/2)} \\
&< \quad 2^n\, \mathrm{VR}\ B_p(\rho^a(p))\, \frac{(r^a_{|R|}(p))^{-n}\, \mathrm{Vol}^{-1} B(r^a_{|R|}(p))}{(\rho^a(p)/2)^{-n}\, \mathrm{Vol}^{-1} B(\rho^a(p)/2)} \\
&< \quad 2^n\, \mathrm{VR}\ B_p(\rho^a(p))\, \frac{a^{-n}\, \mathrm{Vol}^{-1} B(a)}{a^{-n}\, \mathrm{Vol}^0 B(a)}
\end{aligned}$$

□

23.2 F-structures

The constructions of Cheeger-Gromov and Cheeger-Fukaya-Gromov that we have studied were designed for the case of collapse with bounded curvature. In the present situation, we have *collapse with locally bounded curvature*, which follows from 23.1.2. According to the older results, there exist constants $v = v(n)$ so that if a set $U \subset M^n$ is v-collapsed with bounded curvature, in the sense that $p \in U$ implies $B_1(p)$ has $|\mathrm{Rm}| \leq 1$ and $\mathrm{Vol}\ B_1(p) < v$, then

i) (Local structure on a fixed scale) There are numbers $c = c(n)$, $r = r(n)$ so that $p \in U$ implies $B_r(p) \subset T_{3r}(\mathcal{O}_q)$ where $\mathcal{O}_q$ is the orbit through some point q, an so that $\mathcal{O}_q$ has $|II| < c$.

ii) (Invariant metric) There is a metric $\tilde{g}$ on the saturation of U that is invariant under the action of the F-structure.

Moreover, given $\epsilon < 0$, and integer $k \geq 0$, and a number C, we can choose $\eta = \eta(n, \epsilon)$ and choose $\delta = \delta(n, \epsilon, k, C)$ so that

iii) (Diameter of orbits) If $v < \eta$ then the extrinsic diameter of the orbits $\mathcal{O}$ is $< \epsilon$.

iv) (Closeness of the invariant metric) If $v < \delta$ then $|\nabla^i \operatorname{Rm}| < C$ for $i \in \{0, \dots, k\}$, and $|\nabla(\operatorname{Rm} - \widetilde{\operatorname{Rm}})| < \epsilon$ for $i \in \{0, \dots, k-1\}$.

The constructions in the Cheeger-Gromov papers on F-structures require curvature to be small compared to the injectivity radius. However the Cheeger lemma indicates that, on the scale of the curvature radius, volume-collapse and injectivity-radius collapse are equivalent.

Consider the case of collapse on the a-local curvature scale r^a_R, where we let $0 < a < 1$ (which removes the dependence on a in the conclusion of Lemma 23.1.2). We will call an F-structure *a-standard* if its restriction to the ball $B_{r^a_R(p)}(p)$ has the above properties with respect to the rescaled metric $(r^a_R(p))^{-2} g$. It is easy to prove that r^a_R is Lipschitz with Lipschitz constant 1, so we say that r^a_R *varies moderately on its own scale*. Thus, locally, coverings by balls of the form $B_{r^a_R(p)}(p)$ behave roughly like coverings by balls of fixed radius.

Assume U is v-collapsed on the scale a with locally bounded curvature. We indicate briefly how a-standard F-structures can be achieved. Fix a constant $0 < \zeta < 1$ and choose p_i so that

$$\operatorname{dist}(p_i, p_j) \;\leq\; \zeta \, \min\{r^a_R(p_i),\, r^a_R(p_j)\}.$$

Then the balls $B_{2r^a_R(p_i)}(p_i)$ cover U. A variant of the packing lemma shows that this cover has multiplicity bounded by some constant $N(n)$. We can also arrange it so that if the balls $B_{r^a_R(p_i)}(p_i)$ and $B_{r^a_R(p_i)}(p_i)$ overlap, then

$$(1 - \zeta)\, r^a_R(p_i) \;\leq\; r^a_R(p_j) \;\leq\; (1 + \zeta)\, r^a_R(p_j).$$

After scaling so $r^a_R(p_i) = 1$, a slightly smaller subball of $B_{r^a_R(p_i)}(p_i)$ is almost a flat manifold, so supports an F-structure. Since nearby balls have curvature radii that are not too different there is no trouble gluing the F-structures together according to the method describe in a previous lecture. This process can be repeated, until all the F-structures everywhere can be glued together.

23.3 Applying the Chern-Gauss-Bonnet theorem

Recall the Chern-Gauss-Bonnet formula for Einstein 4-manifolds with boundary:

$$\chi(U) \;=\; \int_U |\operatorname{Rm}|^2 \;+\; \int_{\partial U} \mathcal{TP}_\chi$$

where $|\,\mathcal{TP}_\chi\,| \;\le\; C\left(|\,\mathrm{Rm}\,||II_\partial U| \,+\, |II_\partial U|^3\right)$. If U is a domain and $U^{(a)}$, with $a \le 1$, is t-collapsed on the scale a for a sufficiently small t, and if $\int_{U^{(2a)}} |\,\mathrm{Rm}\,|^{\frac{n}{2}} < \delta$ for a sufficiently small δ, then Lemma 23.1.2 implies that $U^{(a)}$ is collapsed on the scale $r^a_{|R|}$, and therefore carries an F-structure. In particular $\chi(U) = 0$. Then

$$\int_U |\,\mathrm{Rm}\,|^2 \;\le\; C\int_{\partial U}\left(|\,\mathrm{Rm}\,||II_\partial U| \,+\, |II_\partial U|^3\right).$$

A domain U can have a boundary of essentially arbitrarily complexity, eg. consider boundary of the Mandelbrot set, which the same Hausdorff dimension as its interior (!), so we gain no information on the interior energy term from this estimation. However it is possible to approximate an arbitrary open set from the outside with a set that has controlled second fundamental form. We turn to this procedure now.

23.4 Approximating domains from the outside

A domain Z can be approximated from the outside with another domain that has controlled second fundamental form, which is controlled by the only by the *local* ambient geometry, namely, the sectional curvature, *not* the injectivity radius. The following theorem is due to Cheeger-Gromov (1985?).

Theorem 23.4.1 (Equivariant good chopping) *Given $K, \Lambda, D < \infty$ there is a number $C = C(n, K, \Lambda, D)$ so that if $Z \subset M$ is a domain with*

- *The domain is diameter bounded: $T_aZ \subset B_q(Da)$ for some $q \in M$*
- *$|\,\mathrm{Ric}\,| \ge -(n-1)\Lambda/a^2$ on $B_q(3Da)$ (the domain is also area bounded)*
- *$|\,\mathrm{Rm}\,| \le K/a^2$ on $A_{0,a}(U)$.*

then there is a domain U with $Z \subset U \subset T_a(Z)$ and

$$|II_{\partial U}| \;\le\; C\,a^{-1}.$$

Further, if Z is invariant under a group of symmetries G of M, then so is U.

23.5 Shape operator of level sets

Let $G : M \to \mathbb{R}$ be a function with regular value c. The unit normals to the level set $N^{n-1} = \{G = c\}$ are $\pm\frac{\nabla G}{|\nabla G|}$. Choosing one of these we have the shape operator (second fundamental form) $\nabla\frac{\nabla G}{|\nabla G|}$. Since

$$\nabla\frac{\nabla G}{|\nabla G|} \;=\; |\nabla G|^{-1}\,\nabla^2 G \,-\, |\nabla G|^{-2}\,\nabla|\nabla G| \otimes \nabla G$$

and since we are restricting to vectors parallel to N, we have

$$II_N \ = \ |\nabla G|^{-1}\nabla^2 G.$$

23.6 Smoothing the distance function

In proving Theorem 23.4.1, a scaling argument allows us to assume $a = 1$. Consider the distance function from a domain $r = \mathrm{dist}(U, \cdot)$. We have $|\nabla r| = 1$ but no information on the smoothness of $\nabla^2 r$. Of course it is possible to smooth out $\nabla^2 r$. The question is whether control from *below* can be maintained over $|\nabla r|$. To obtain the smoothing of $\nabla^2 r$ one can solve the Laplace equation $\triangle \hat{r} = 0$ with $\hat{r} = 1$ on ∂U and $\hat{r} = 0$ on $\partial T_a(U)$.

Put $d = \min\{\pi/\sqrt{K},\, 1/2\}$ and let $\{p_i\} \subset \partial T_{\frac{1}{2}}(U)$ be a maximal $\frac{d}{4}$-separated set. Then the balls $B_i = B_{p_i}(d/2)$ cover the annulus $A_{\frac{1}{2}-\frac{d}{4},\frac{1}{2}+\frac{d}{4}}(Z)$, and the balls $\frac{1}{4}B_i = B_{p_i}(d/8)$ are disjoint. Gromov's packing argument gives an upper bound on $\sharp\{B_i\}$, that depends on D, K, and Λ.

There are local homeomorphisms $\pi_i : \widetilde{B}_i \to B_i$ where $\widetilde{B}_i$ is a Euclidean ball with the pullback metric. On each $\widetilde{B}_i$ exists the pullback function, still denoted $\hat{r}$. We have the following theorem regarding this pullback function.

Proposition 23.6.1 (Yomdin) *Given a domain U with a Riemannian metric, and numbers $\eta > 0$, $N, K < \infty$, there is a number $T = T(n, U, \eta, N, K) > 0$ so that if $|sec_U| < K$ and a function $f : U \to [0, 1]$ has $C^{1,\alpha}(f) < N$, then the open set $[0, 1] - f(\{x \in U \,|\, |\nabla f(x)| > T\})$ is η-dense in $[0, 1]$.*

<u>*Pf*</u>

Assume there is no such T. Then there is a sequence $T_i \searrow 0$ and functions $f : U \to \mathbb{R}$ with $C^{1,\alpha}(f) < N$ but so that $\{x \in U \,|\, |\nabla f_i(x)| < T_i\}$ has an interval of length $\eta/2$. Taking a limit along a subsequence, we get a function f_∞ with $C^{1,\alpha}(f_\infty) < N$, and a $C^{1,\alpha}$-metric on U, but so that $\{x \in U \,|\, |\nabla f_\infty(x)| = 0\}$ has an interval of length $\eta/2$. By Sard's theorem, this is a contradiction. □

Chapter 24

Einstein Manifolds VI

May 4, 2010

24.1 Maximal functions and L^α estimates

24.2 Maximal functions and L^α estimates

Recall the Littlewood maximal function

$$M_f(p) \;=\; \sup_{r>0} \frac{1}{\mathrm{Vol}\ B_p(r)} \int_{B_p(r)} |f|,$$

and its localized version

$$M_f^s(p) \;=\; \sup_{0<r<s} \frac{1}{\mathrm{Vol}\ B_p(r)} \int_{B_p(r)} |f|.$$

We already have a lower bound on $r^s_{|R|}(p)$ in terms of $\rho^s(p)$. The following proposition gives a lower bound in terms of the maximal function.

Lemma 24.2.1 (Weak curvature radius estimate) *There exists some $C = C(n)$ so that whenever $s > 0$ then*

$$\left(r^s_{|R|}(p)\right)^{-k} \;\leq\; C\left(s^{-k} \;+\; \left(M^2_{|\,\mathrm{Rm}\,|^{\frac{n}{2}}}(p)\right)^{\frac{k}{n}}\right)$$

Pf

Recall that $r^s_{|R|}(p) \;\geq\; \frac{1}{2}\rho^s(p)$. Then if $\rho^s(p) = s$ then $r^s_{|R|}(p) \;\geq\; \frac{s}{2}$ and the conclusion holds. If $\rho^s(p) < s$ then

$$(\rho^s(p))^n \;=\; \epsilon_0^{-1}\frac{1}{\text{Vol } B_{\rho^s(p)}}\int_{B_{\rho^s(p)}}|\operatorname{Rm}|^2 \;\leq\; \epsilon_0^{-1}M^s_{|\operatorname{Rm}|^{\frac{n}{2}}}(p).$$

Thus $r^s_{|R|}(p) \;\leq\; \epsilon_0^{-\frac{1}{n}}\left(M^s_{|\operatorname{Rm}|^{\frac{n}{2}}}\right)^{\frac{1}{n}}$ and again the conclusion holds. □

To proceed we require more information on maximal functions. We recall some standard results.

Lemma 24.2.2 (The maximal theorem) *Assume $K \subset M^n$ is a subset of finite measure, and assume the following doubling condition holds for balls in K: $\operatorname{Vol} B_{2r}(p) \leq C_D 2^n \operatorname{Vol} B_r(p)$ whenever $p \in K$ and $B_p(r/2) \subset K$. Let $f \in L^1(K^{(4s)})$ and put $K_\lambda = \{\, p \in K \mid M^s_f(p) > \lambda \,\}$. Then*

$$\text{Vol } K_\lambda \;\leq\; C_D^2 4^n \lambda^{-1}\|f\|_{L^1(K^{(s)})}.$$

<u>*Pf*</u>

Assuming that $p \in K_\lambda$, then by the definition of the maximal function there is a number $r_p < s$ so that

$$\lambda \;\leq\; \frac{1}{\text{Vol } B_{r_p}(p)}\int_{B_{r_p}(p)}|f|.$$

Let p_i ne a set of points so that the $B_{r_{p_i}}(p_i)$ are all disjoint, but so that $K_\lambda \subset \bigcup_i B_{4r_{p_i}}(p_i)$. Then

$$\begin{aligned}
\text{Vol } K_\lambda \;&\leq\; \sum_i \text{Vol } B_{p_i}(4r) \;\leq\; C_D^2 4^n \sum_i \text{Vol } B_{r_{p_i}}(p_i) \\
&\leq\; C_D^2 4^n \lambda^{-1}\sum_i \int_{B_{p_i}(r_{p_i})}|f| \\
&\leq\; C_D^2 4^n \lambda^{-1}\|f\|_{L^1(K^{(s)})}.
\end{aligned}$$

□

Lemma 24.2.3 *If $f \geq 0$ is a function in $L^k(\Omega)$ and $\Omega_\lambda = \{f > \lambda\}$ are the superlevel sets, then*

$$\int_\Omega f^k \;\leq\; \left(\int_0^\infty |\Omega_\lambda|^{\frac{1}{k}}\right)^k.$$

<u>*Pf*</u>

We have

$$\begin{aligned}
\int_\Omega f^k \;&=\; \int_\Omega\int_0^{f(p)} ks^{k-1}ds\,d\mu(p) \;=\; \int_0^\infty ks^{k-1}\int_{\Omega_s} \\
&=\; \int_0^\infty ks^{k-1}|\Omega_s|\,ds
\end{aligned}$$

But because $|\Omega_s|$ is decreasing

$$\begin{aligned} s^{k-1}|\Omega_s|^{\frac{k-1}{k}} &\leq \left(\int_0^s |\Omega|^{\frac{1}{k}} ds\right)^{k-1} \\ ks^{k-1} &\leq k|\Omega_s|^{\frac{1}{k}} \left(\int_0^s |\Omega_\lambda|^{\frac{1}{k}} d\lambda\right)^{k-1} = \frac{d}{ds}\left(\int_0^s |\Omega_\lambda|^{\frac{1}{k}}\right)^k. \end{aligned}$$

The result follows from integrating both sides. □

Lemma 24.2.4 *Assume that $K \subset M^n$ is any compact domain so that $K^{(4s)}$ has a doubling constant C_D, and assume that $\alpha \in (0,1)$. Then*

$$\left(\frac{1}{|K^{(a)}|}\int_K \left(M_f^a\right)^\alpha\right)^{\frac{1}{\alpha}} \leq C(n,\alpha,C_D)\frac{1}{|K^{(a)}|}\int_{K^{(a)}} |f|.$$

<u>*Pf*</u>

The previous lemma gives

$$\begin{aligned} \left(\int_K \left(M_f^a\right)^\alpha\right)^{\frac{1}{\alpha}} &\leq \int_0^\infty |K_\lambda|^{\frac{1}{\alpha}} d\lambda \\ &= \int_0^B |K_\lambda|^{\frac{1}{\alpha}} d\lambda + \int_B^\infty |K_\lambda|^{\frac{1}{\alpha}} d\lambda \\ &\leq B|K|^{\frac{1}{\alpha}} + C_D^{\frac{2}{\alpha}} 4^{\frac{n}{\alpha}} \|f\|_{L^1(K^{(a)})}^{\frac{1}{\alpha}} \int_B^\infty \lambda^{-\frac{1}{\alpha}} d\lambda \\ \left(\frac{1}{|K^{(a)}|}\int_K \left(M_f^a\right)^\alpha\right)^{\frac{1}{\alpha}} &\leq B + \frac{\alpha}{1-\alpha} C_D^{\frac{2}{\alpha}} 4^{\frac{n}{\alpha}} |K^{(a)}|^{-\frac{1}{\alpha}} \|f\|_{L^1(K^{(a)})}^{\frac{1}{\alpha}} B^{\frac{\alpha-1}{\alpha}}. \end{aligned}$$

Now putting $B = \frac{1}{|K|^{(a)}}\int_{K^{(a)}} |f|$ we get

$$\left(\frac{1}{|K^{(a)}|}\int_K \left(M_f^a\right)^\alpha\right)^{\frac{1}{\alpha}} \leq \left(1 + \frac{\alpha}{1-\alpha} C_D^{\frac{2}{\alpha}} 4^{\frac{n}{\alpha}}\right) \frac{1}{|K^{(a)}|}\int_{K^{(a)}} |f|.$$

□

24.3 The Key Estimate

Again we assume $U^{(a)}$ is collapsed on the scale a, and has small energy.

We can use the good chopping theorem, localized and applied at the curvature scale, to find an open set Z with $U \subset Z \subset U^{(a)}$ and $|II_{\partial Z}| < C\left(r^a_{|R|}\right)^{-1}$. With $|\operatorname{Rm}| < 4\left(r^a_{|R|}\right)^{-2}$

and $|II_{\partial Z}| < C\left(r^a_{|R|}\right)^{-1}$, we have

$$\int_U |\,\mathrm{Rm}\,|^2 \;\; \leq \;\; \int_{\partial Z} |\,\mathcal{TP}\,| \;\leq\; C\int_{\partial Z}\left(r^a_{|R|}\right)^{-3}.$$

An averaging procedure lets us write

$$\int_U |\,\mathrm{Rm}\,|^2 \;\; \leq \;\; Ca^{-1}\int_{U^{(\frac{3}{4}a)}-U^{(\frac{1}{4}a)}}\left(r^a_{|R|}\right)^{-3}.$$

Using Lemma 24.2.1 we have

$$\int_U |\,\mathrm{Rm}\,|^2 \;\; \leq \;\; Ca^{-4}|U^{(\frac{3}{4}a)}-U^{(\frac{1}{4}a)}| \,+\, Ca^{-1}\int_{U^{(\frac{3}{4}a)}-U^{(\frac{1}{4}a)}}\left(M^a_{|\,\mathrm{Rm}\,|^2}\right)^{\frac{3}{4}}$$

and Lemma 24.2.4 gives

$$\int_U |\,\mathrm{Rm}\,|^2 \;\; \leq \;\; Ca^{-4}|U^{(\frac{3}{4}a)}-U^{(\frac{1}{4}a)}| \,+\, Ca^{-1}|U^{(\frac{3}{4}a)}-U^{(\frac{1}{4}a)}|^{\frac{1}{4}}\left(\int_{U^{(\frac{3}{4}a)}-U^{(\frac{1}{4}a)}}|\,\mathrm{Rm}\,|^2\right)^{\frac{3}{4}}.$$

The exponent improvement permits an iteration process to go through. Specifically we can put $a_i = \int_{U^{(a-2^{-i}a)}-U^{(2^{-i}a)}}|\,\mathrm{Rm}\,|^2$ and $b_i = 2^{-i}a^{-4}|U^{(a)}-U| \;=\; 2^{-i}b_0$, and we get

$$\begin{aligned}
a_i \;\; &\leq \;\; C\left(b_i \,+\, b_i^{\frac{1}{4}}a_{i+1}^{\frac{3}{4}}\right)\\
&\leq \;\; \max\left\{2Cb_i,\, 2Cb_i^{\frac{1}{4}}a_{i+1}^{\frac{3}{4}}\right\}\\
&\leq \;\; \max\left\{2Cb_i,\, 2Cb_i^{\frac{1}{4}}\max\left\{2Cb_{i+1},\, 2Cb_{i+1}^{\frac{1}{4}}a_{i+2}^{\frac{3}{4}}\right\}^{\frac{3}{4}}\right\}\\
&\leq \;\; \max\left\{2Cb_i,\, 2Cb_i^{\frac{1}{4}}(2Cb_{i+1})^{\frac{3}{4}},\, 2Cb_i^{\frac{1}{4}}\left(2Cb_{i+1}^{\frac{1}{4}}a_{i+2}^{\frac{3}{4}}\right)^{\frac{3}{4}}\right\}\\
&\leq \;\; \max\left\{(2C)^{1+\frac{3}{4}}2^{\frac{3}{4}}b_i,\, (2C)^{1+\frac{3}{4}}b_i^{\frac{1}{4}\left(1+\frac{3}{4}\right)}2^{\frac{1}{4}\left(1+\frac{3}{4}\right)}(a_{i+2})^{\left(\frac{3}{4}\right)^2}\right\}.
\end{aligned}$$

Evidently an iteration argument can proceed. We get

$$a_i \;\; \leq \;\; \max\left\{(2C)^{1+\frac{3}{4}+\cdots+\left(\frac{3}{4}\right)^j}2^{1+\frac{3}{4}+\cdots+\left(\frac{3}{4}\right)^j}b_i,\, (2C)^{1+\frac{3}{4}+\cdots+\left(\frac{3}{4}\right)^j}b_i^{\frac{1}{4}\left(1+\frac{3}{4}+\cdots+\left(\frac{3}{4}\right)^j\right)}2^{\frac{1}{4}\left(1+\frac{3}{4}+\cdots+\left(\frac{3}{4}\right)^j\right)}(a_{i+j})^{\left(\frac{3}{4}\right)^j}\right\}.$$

Since a_k is bounded, we get $\lim_{j\to\infty} a_{i+j}^{(3/4)^j} \leq 1$. Thus letting $j \to \infty$ we get

$$a_i \;\; \leq \;\; C\,b_i.$$

In particular this applies to $a_2 = \int_{U^{(\frac{3}{4}a)}-U^{(\frac{1}{4}a)}}|\,\mathrm{Rm}\,|^2$, so that

$$\begin{aligned}
\int_U |\,\mathrm{Rm}\,|^2 \;\; &\leq \;\; Ca^{-4}|U^{(\frac{3}{4}a)}-U^{(\frac{1}{4}a)}| \,+\, Ca^{-1}|U^{(\frac{3}{4}a)}-U^{(\frac{1}{4}a)}|^{\frac{1}{4}}\left(\int_{U^{(\frac{3}{4}a)}-U^{(\frac{1}{4}a)}}|\,\mathrm{Rm}\,|^2\right)^{\frac{3}{4}}\\
&\leq \;\; Ca^{-4}|U^{(a)}-U|.
\end{aligned}$$

This is Cheeger-Tian's "Key Estimate."

Chapter 25

Einstein Manifolds VII - Epsilon regularity

May 6, 2010

25.1 Energy ratio improvement

We have called scale-invariant quantity $\frac{1}{\mathrm{VR}\ B_p(r)}\int_{B_p(r)}|\,\mathrm{Rm}\,|^{\frac{n}{2}}$ the "energy ratio." It is convenient to modify this, and consider the quantity

$$\frac{\mathrm{Vol}^{\lambda}\,B(r)}{\mathrm{Vol}\;B_p(r)}\int_{B_p(r)}|\,\mathrm{Rm}\,|^{\frac{n}{2}}.$$

Since we have $\mathrm{Ric}\geq\lambda g$, this quantity is more useful in the use of relative volume comparison. If we restrict ourselves to $a\leq 1$ then these quantities are equivalent.

Lemma 25.1.1 *Assume M^n is an Einstein manifold, $r\leq 1$, and $\int_{B_p(r)}|\,\mathrm{Rm}\,|^{\frac{n}{2}}<\delta$. Then there exist numbers $C<\infty$, $\eta>0$ so that either*

$$\frac{\mathrm{Vol}^{\lambda}\,B(r/2)}{\mathrm{Vol}\;B_p(r/2)}\int_{B_p(r/2)}|\,\mathrm{Rm}\,|^2\;\leq\;(1\;-\;\eta)\,\frac{\mathrm{Vol}^{\lambda}\,B(r/2)}{\mathrm{Vol}\;B_p(r)}\int_{B_p(r)}|\,\mathrm{Rm}\,|^2 \qquad (25.1)$$

or else the annulus $B_p(5r/8)-B_p(3r/8)$ has

$$\begin{aligned}|\,\mathrm{Rm}\,| \quad &< \quad Cr^{-2}\sqrt{\eta}\\ \frac{\mathrm{Vol}\;B_p(r)}{\mathrm{Vol}\;B_p(r/2)} \quad &\geq \quad (1-\eta)\frac{\mathrm{Vol}^{\lambda}\,B(r)}{\mathrm{Vol}^{\lambda}\,B(r/2)}.\end{aligned}$$

Pf

If (25.1) does not hold, then

$$\begin{aligned}\frac{\mathrm{Vol}^\lambda\, B(r/2)}{\mathrm{Vol}\ B_p(r/2)}\int_{B_p(r/2)}|\,\mathrm{Rm}\,|^2 &\geq (1\,-\,\eta)\,\frac{\mathrm{Vol}^\lambda\, B(r/2)}{\mathrm{Vol}\ B_p(r)}\int_{B_p(r)}|\,\mathrm{Rm}\,|^2\\ &\geq (1\,-\,\eta)\,\frac{\mathrm{Vol}^\lambda\, B(r/2)}{\mathrm{Vol}\ B_p(r)}\int_{B_p(r/2)}|\,\mathrm{Rm}\,|^2,\end{aligned}$$

and we have $\frac{\mathrm{Vol}^\lambda\, B(r/2)}{\mathrm{Vol}\ B_p(r/2)} \geq (1-\eta)\frac{\mathrm{Vol}^\lambda\, B(r/2)}{\mathrm{Vol}\ B_p(r)}$. On the other hand

$$\int_{B_p(r)-B_p(r/2)}|\,\mathrm{Rm}\,|^2 \;\leq\; \left(\frac{1}{1-\eta}\frac{\mathrm{Vol}^\lambda\, B(r/2)}{\mathrm{Vol}^\lambda\, B(r)}\frac{\mathrm{Vol}\ B_p(r)}{\mathrm{Vol}\ B_p(r/2)}\;-\;1\right)\int_{B_p(r/2)}|\,\mathrm{Rm}\,|^2.$$

Bishop-Gromov volume comparison gives $\frac{\mathrm{Vol}^\lambda\, B(r/2)}{\mathrm{Vol}^\lambda\, B(r)}\frac{\mathrm{Vol}\ B_p(r)}{\mathrm{Vol}\ B_p(r/2)} \leq 1$, so therefore

$$\int_{B_p(r)-B_p(r/2)}|\,\mathrm{Rm}\,|^2 \;\leq\; \frac{\eta}{1-\eta}\int_{B_p(r/2)}|\,\mathrm{Rm}\,|^2.$$

The Key Estimate now gives

$$\int_{B_p(r)-B_p(r/2)}|\,\mathrm{Rm}\,|^2 \;\leq\; \frac{\eta}{1-\eta}Cr^{-4}\,\left(\mathrm{Vol}\ B_p(r)\,-\,\mathrm{Vol}\ B_p(r/2)\right).$$

Now let $q \in B_p(5r/8) - B_p(3r/8)$, so that $B_q(r/8) \subset B_p(r) - B_p(r/2)$. We have

$$\begin{aligned}\int_{B_q(r/8)}|\,\mathrm{Rm}\,|^2 &\leq \int_{B_p(r)-B_p(r/2)}|\,\mathrm{Rm}\,|^2\\ &\leq \frac{\eta}{1-\eta}Cr^{-4}\,\left(\mathrm{Vol}\ B_p(r)\,-\,\mathrm{Vol}\ B_p(r/2)\right)\\ &\leq \frac{\eta}{1-\eta}Cr^{-4}\ \mathrm{Vol}\ B_q(2r)\\ &\leq \frac{\eta}{1-\eta}Cr^{-4}\ \mathrm{Vol}\ B_q(r/8)\frac{\mathrm{Vol}^\lambda\, B_q(2r)}{\mathrm{Vol}^\lambda\, B_q(r/8)}\end{aligned}$$

so that

$$\frac{\mathrm{Vol}^\lambda\, B(r/8)}{\mathrm{Vol}\ B_q(r/8)}\int_{B_q(r/8)}|\,\mathrm{Rm}\,|^2 \quad\leq\quad \frac{\eta}{1-\eta}Cr^{-4}\mathrm{Vol}^\lambda B_q(2r).$$

With $r \leq 1$ we have that there exists a C so that

$$\frac{\mathrm{Vol}^\lambda\, B(r/8)}{\mathrm{Vol}\ B_q(r/8)}\int_{B_q(r/8)}|\,\mathrm{Rm}\,|^2 \quad\leq\quad \frac{\eta}{1-\eta}C.$$

Now if η is chosen small enough that $C\eta/(1-\eta) < \epsilon_0$, then ϵ-regularity holds and we get

$$|\mathrm{Rm}_q| \;\leq\; C\,r^{-2}\sqrt{\eta}$$

□

Lemma 25.1.2 *The second alternative in Lemma 25.1.1 does not hold.*

Pf

The small curvature and almost-volume annulus together imply the existence of a Cheeger-Colding function $\hat{r}$ that has the following properties

$$\begin{aligned}
&\triangle \hat{r}^2 \;=\; 8 \\
&|\hat{r} \,-\, r| \;\leq\; \Phi \\
&\frac{1}{|\hat{r}^{-1}(a)|}\int_{\hat{r}^{-1}(a)} |\nabla\hat{r} \,-\, \nabla r|^2 \;\leq\; \Phi \\
&|\nabla\hat{r}| \;\leq\; C \\
&\left|1 \,-\, \frac{|\hat{r}^{-1}(a)|}{|\partial B_p(a)|}\right| \;\leq\; \Phi \\
&\frac{1}{|\hat{r}^{-1}(a)|}\int_{\hat{r}^{-1}(a)} \left| II_{\hat{r}^{-1}(a)} \,-\, \frac{1}{\hat{r}} g_{\hat{r}^{-1}(a)} \otimes \nabla\hat{r}\right| \;\leq\; \Phi.
\end{aligned}$$

for some $\Phi = \Phi(\eta)$ where $\lim_{\eta\to 0}\Phi = 0$. We can pass to the universal covering space, where the injectivity radius is bounded and we can take a limit. On this space the injectivity radius is bounded, so it is possible to take a limit as $\eta \to 0$. On the limit space the function $\hat{r}$ has $\nabla^2 r = \frac{1}{\hat{r}} g$, so the limit is a warped product with level sets of $\hat{r}$ being space forms. This gives $C^{1,\alpha}$-convergence of $\hat{r}$.

Therefore $\hat{r}^{-1}$ converges in the *pointwise* sense to a space form. Thus the annulus has (almost) the metric structure of an annulus in a Euclidean cone.

Now consider again the Chern-Gauss-Bonnet theorem

$$\chi(B_p(3r/4)) \;=\; \int |\operatorname{Rm}|^2 \;+\; \int_{\partial B_p(3r/4)} \mathcal{TP}_\chi.$$

The boundary term converges to the Euclidean boundary term, which is *positive*. Since the left-hand side is negative due to the F-structure, we have a contradiction. □

Theorem 25.1.3 (ϵ-regularity) *If $r \leq 1$ and $\int_{B_p(r)} |\operatorname{Rm}|^2 \;\leq\; \delta$, then for some $\mu > 0$*

$$\sup_{B_p(\mu r)} |\operatorname{Rm}|^2 \;\leq\; C r^{-2}.$$

Pf

The Key Estimate gives

$$\begin{aligned}
&\int_{B_p(r/2)} |\operatorname{Rm}|^2 \;\leq\; C r^{-4}\, |\mathrm{Vol}\; B_p(r) - \mathrm{Vol}\; B_p(r/2)| \\
&\frac{\mathrm{Vol}^\lambda\, B(r/2)}{\mathrm{Vol}\; B_p(r/2)} \int_{B_p(r/2)} |\operatorname{Rm}|^2 \;\leq\; C
\end{aligned}$$

for some C. Lemmas 25.1.1 and 25.1.2 give

$$\frac{\mathrm{Vol}^{\lambda}\, B(r2^{-k-1})}{\mathrm{Vol}\, B_p(r2^{-k-1})}\int_{B_p(r/2)} |\,\mathrm{Rm}\,|^2 \;\leq\; C\eta^k.$$

Choosing $k > \frac{\log(\epsilon_0/C)}{\log(\eta)}$ gives

$$\frac{\mathrm{Vol}^{\lambda}\, B(r2^{-k-1})}{\mathrm{Vol}\, B_p(r2^{-k-1})}\int_{B_p(r/2)} |\,\mathrm{Rm}\,|^2 \;\leq\; \epsilon_0,$$

whereupon standard ϵ-regularity goes through. □