Lecture note on Lie Algebras

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These notes are purely expository; all credit for the ideas and methods is due to others. As a warning, these being daily notes composed in an hour or two before class each day, they are prone to errors, typos, and other errata. Each “chapter” represents a single day’s lecture.

Part I of these lecture notes almost mostly consists of notes on the text ”Introduction to Lie Algebras and Representation Theory” by James E. Humphreys. Notable exceptions are our look into the interaction of Lie algebra theory with the quantum theory in Chapter 10, and the exercises in Chapter 14. Unfortuantely many of the notes from the end of Fall 2012 have been lost; at some point I may reproduce those notes from outlines that I still posess.

Part II was taken from a variety of sources.

Part I

Fall 2012

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Chapter 1

Basic Definitions and Examples of Lie Algebras

September 6, 2012

1.1 Definition

A *Lie algebra* $\mathfrak{l}$ is a vector space V over a base field $\mathbb{F}$, along with an operation $[\cdot, \cdot] : V \times V \to V$ called the *bracket* or *commutator* that satisfies the following conditions:

- Bilinearity: $\alpha[x, y] = [\alpha x, y] = [x, \alpha y]$ for $\alpha \in \mathbb{F}$ and $x, y \in V$
- Antisymmetry: $[x,\, y] \;=\; -[y,\, x]$ for $x, y \in V$
- The Jacobi Identity: $[x,\, [y,\, z]] \,+\, [y,\, [z,\, x]] \,+\, [z,\, [x,\, y]] \;=\; 0$ for $x, y, z \in V$.

This definition works for any base field except those of characteristic 2, in which case the second condition is replaced by $[x, x] = 0$, which is then not equivalent to $[x, y] = -[y, x]$. However most theorems required algebraic completeness, which will generally be assumed. We will also generally ignore the case of finite characteristic, which sometimes requires additional considerations. Thus realistically we are talking about $\mathbb{F} = \mathbb{C}$. In addition, we almost always require that V be finite dimensional.

It should be noted that the Jacobi identity is formally identical to the Leibnitz rule.

A brute-force construction of an arbitrary Lie algebra may begin with selecting a finite-dimensional vector field V, a basis $\{e_i\}_{i=1}^n$, and n^3 many constants $c_{ij}^k \in \mathbb{F}$ subject to the relations

- $c_{ij}^k = -c_{ji}^k$
- $\sum_{k=1}^n \left(c_{is}^u c_{jk}^s \,+\, c_{js}^u c_{ki}^s \,+\, c_{ks}^u c_{ij}^s \right) \;=\; 0.$

One then defines the brackets $[e_i, e_j] = \sum_{k=1}^n c_{ij}^k e_k$ and extends by linearity.

1.2 Non-Fundamental Examples

1.2.1 The abelian case

If V is any vector space with origin o, it has the structure of a Lie algebra where we define

$$[x,\, y] \;=\; o \tag{1.1}$$

for all $x, y \in V$. Any such example is called an abelian Lie algebra.

1.2.2 The cross product

In high school we had the cross product on $\mathbb{R}^3$, given by

$$\vec{X} \times \vec{Y} \;=\; \sin(\theta)|\vec{X}||\vec{Y}| \,\cdot\, \hat{n}_{\vec{X}\vec{Y}} \tag{1.2}$$

where $\hat{n}_{\vec{X}\vec{Y}}$ is the unit normal to $\vec{X}$ and $\vec{Y}$ chosen by the right-hand rule, and θ is the angle between the vectors. Orientation (the right-hand rule) requires that

$$\vec{X} \times \vec{Y} \;=\; -\vec{Y} \times \vec{X} \tag{1.3}$$

and a simple check shows that

$$\vec{X} \times \left(\vec{Y} \times \vec{Z}\right) \,+\, \vec{Y} \times \left(\vec{Z} \times \vec{X}\right) \,+\, \vec{Z} \times \left(\vec{X} \times \vec{Y}\right) \;=\; 0. \tag{1.4}$$

1.2.3 The Heisenberg algebra

In 1-dimensional quantum mechanics, the state of a system is given by an L^2 function $f : \mathbb{R} \to \mathbb{R}$, denoted $f(q)$. The collection of all such states is the Hilbert space $\mathcal{H} = L^2(\mathbb{R})$. The coordinate q can itself be considered an operator $q : \mathcal{H} \to \mathcal{H}$, which acts by multiplication: $f(q) \mapsto qf(q)$. Thinking for a moment like a physicist and ingoring the fact that derivatives of L^2 functions do not exist and might not belong to L^2 if they did, a second operator $p : \mathcal{H} \to \mathcal{H}$, called the momentum operator, can be given by

$$p\,(f) \;=\; \sqrt{-1}\hbar\frac{\partial f}{\partial q}. \tag{1.5}$$

Letting $[\cdot,\cdot]$ be the usual commutator, the action of $[p,q]$ on $f \in \mathcal{H}$ is

$$[p,\, q]\; f \;=\; \sqrt{-1}\,\hbar\,\frac{\partial}{\partial q}\,(q\,f) \,-\, q\left(\sqrt{-1}\,\hbar\,\frac{\partial f}{\partial q}\right) \tag{1.6}$$

$$=\; \sqrt{-1}\,\hbar\, f, \tag{1.7}$$

or in other words, multiplication by the constant $\sqrt{-1}\,\hbar$. The *Heisenberg algebra* is the Lie algebra on

$$V \;=\; span_{\mathbb{C}}\{p,\, q,\, \sqrt{-1}\,\hbar\} \tag{1.8}$$

with this bracket operation.

Of course this algebra can be defined abstractly by $V = span_{\mathbb{C}}\{X, Y, Z\}$ where

$$[X,\, Y] \;=\; Z \qquad [X,\, Z] \;=\; 0 \qquad [Y,\, Z] \;=\; 0. \tag{1.9}$$

The Heisenberg algebra is isomorphic to the concrete example given by

$$X = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad Y = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \qquad Z = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{1.10}$$

and $V = span_{\mathbb{C}}\{X, Y, Z\}$, with the bracket given by the usual matrix commutator.

1.2.4 The angular momentum operators

In 3D quantum mechanics, we have three position operators q^1, q^2, q^2 and three momentum operators $p_i = \sqrt{-1}\hbar\frac{\partial}{\partial q^i}$, with the usual commutators. From these other useful operators are defined, such as l_x, l_y, l_z, whose eigenvalues record a wavefunction's angular momenta about the the respective axes. They are defined by

$$l_x \;=\; q^2 p_3 \,-\, q^3 p_2 \qquad l_y \;=\; q^3 p_1 \,-\, q^1 p_3 \qquad l_z \;=\; q^1 p_2 \,-\, q^2 p_1. \tag{1.11}$$

One easily chechs the commutator relations

$$[l_x,\, l_y] \;=\; \sqrt{-1}\,\hbar\, l_z \qquad [l_y,\, l_z] \;=\; \sqrt{-1}\,\hbar\, l_x \qquad [l_z,\, l_x] \;=\; \sqrt{-1}\,\hbar\, l_y. \tag{1.12}$$

Under the basis $X = \frac{1}{\sqrt{-1}\,\hbar} l_x, Y = \frac{1}{\sqrt{-1}\,\hbar} l_y, Z = \frac{1}{\sqrt{-1}\,\hbar} l_z$, we have

$$[X,\, Y] \;=\; Z \qquad [Y,\, Z] \;=\; X \qquad [Z,\, X] \;=\; Y. \tag{1.13}$$

This is just the cross product algebra (from above), and is the same as the Lie algebras $\mathfrak{so}(3)$ and $\mathfrak{su}(2)$ (below).

1.2.5 Lie Groups

A Lie group is a differentiable manifold G along with a group structure so that the group operation (multiplication and inversion) are differentiable. Any Lie group M has an associated Lie algebra $\}$, given by derivations at the identity $I \in G$. Specifically, any element a of the tangent space $T_I G$ at the identity can be represented by a path $A : (-\epsilon, \epsilon) \to M$ with $A(0) = I$. Given $a, b \in T_I G$, we define their bracket $[a,\, b]$ to be the vector given by one half the second derivative of the path

$$t \;\mapsto\; A(t)\, B(t)\, A(t)^{-1}\, B(t)^{-1}. \tag{1.14}$$

If G is an Abelian group, then clearly $\mathfrak{g}$ is an abelian Lie algebra.

1.2.6 Linear Lie Groups

If G is a linear Lie group, meaning a subgroup of some $L(V)$, then its Lie algebra $\mathfrak{g}$, defined before as derivations at the identity, will have its bracket given precisely by the usual commutator operation on matrices. Indeed, letting the paths $A(t), B(t)$ represent the vectors a, b, then using the definition we compute

$$\begin{aligned}
[a,\, b] \;&=\; \frac{1}{2}\frac{d^2}{dt^2}\Big|_{t=0} A(t)B(t)A(t)^{-1}B(t)^{-1} && (1.15)\\
&=\; \left(\frac{dA}{dt}\frac{dB}{dt} \;+\; \frac{dA^{-1}}{dt}\frac{dB^{-1}}{dt} \;+\; \frac{dA}{dt}\frac{dB^{-1}}{dt} \;+\; \frac{dB}{dt}\frac{dA^{-1}}{dt}\right)\Big|_{t=0} && (1.16)\\
&=\; \dot{A}(0)\dot{B}(0) \;-\; \dot{B}(0)\dot{A}(0) \;=\; ab - ba && (1.17)
\end{aligned}$$

where we have abbreviated $\dot{A}(\tau) = \frac{d}{dt}\big|_{t=\tau} A(t)$.

Example. Consider the *special Linear group* $SL(n, \mathbb{C})$, defined to be group of matrices with unit determinant. If $a(t)$ is a path in $SL(n, \mathbb{C})$ with $det(a(t)) = 1$ and $a(0) = I$, then

$$0 \;=\; \frac{d}{dt}\Big|_{t=0} \det(a(t)) \;=\; \text{Trace}\left(\frac{da}{dt}\Big|_{t=0}\right) \tag{1.18}$$

so the associated Lie algebra $\mathfrak{g}$ is the *special linear algebra*, or $\mathfrak{sl}(n, \mathbb{C})$, which consists of the trace-free matrices. (Of course we only proved containment—actual equality comes from exponentiation along with $det(e^A) = e^{Tr(A)}$.)

Example. Consider the *orthogonal group*, $O(n)$, consisting of $n \times n$ matrices A such that $A^T A = I$. Taking derivatives at the origin, we see that if $A(t)$ is a path through $I \in O(n)$ with $a = \dot{A}(0)$, we have

$$0 \;=\; \frac{d}{dt}\Big|_{t=0} \left(A(t)^T A(t)\right) \;=\; a^T \;+\; a. \tag{1.19}$$

The Lie algebra therefore associated with $O(n)$ is the *orthogonal algebra* $\mathfrak{o}(n)$, consisting of antisymetric $n \times n$ matrices.

Example. The third main examples is the symplectic group on V. Assume V has dimension $2n$, and, after choosing a basis, let

$$J \;=\; \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}. \tag{1.20}$$

The symplectic group $Sp(2n, \mathbb{C})$ consists of matrices A so that $A^T J A = I_{2n}$. Taking derivatives, we see that if $A(t)$ is a path in $Sp(2n, \mathbb{C})$ with $A(0) = I_{2n}$ and $\dot{A}(0) = a$, then

$$0 \;=\; \frac{d}{dt}\Big|_{t=0} \left(A^T J A\right) \;=\; a^T J \,+\, J\,a. \tag{1.21}$$

The *symplectic algebra* $\mathfrak{sl}(2n, \mathbb{C})$ is the set of complex matrices a so that $a^T J = -Ja$. (In a bit of unfortunate notation, the groups $Sp(2n, \mathbb{C})$ are not the same as the groups $Sp(n)$.)

1.3 The Fundamental Examples

1.3.1 The general linear algebra, $\mathfrak{gl}$

Let V be any vector space, with $L(V)$ its linear group ($End(V)$ is equivalent notation). We know $L(V)$ is a vector space, and has the structure of an associative algebra under the usual operation of composition. It also carries the structure of a Lie algebra, denoted $\mathfrak{gl}(V)$, where the bracket is the usual commutator

$$[x,\, y] \;=\; x \circ y \,-\, y \circ x. \tag{1.22}$$

(Exercise: Verify the Jacobi identity). The Lie algebra $\mathfrak{gl}(V)$ should not be confused with the general linear group $GL(V)$ (the subgroup of $L(V)$ of invertible transformations); in particular $GL(V)$ is not a vector space so cannot be a Lie algebra.

Any subspace of any $gl(V)$ that is closed under the commutator operation is known as a linear Lie algebra.

1.3.2 Series A, B, C, and D

Cartan's notation for the special linear algebras was A_l, which is defined to be simply $\mathfrak{sl}(l+1, \mathbb{C})$. Likewise, the C-series algebras are precisely the symplectic algebras defined above: $C_l = \mathfrak{sp}(2l, \mathbb{C})$. For reasons that are certainly not clear at present, the orthogonal algebras are divided into two series, with the B-series being the odd dimensional and the D-series being the even dimensional orthogonal algebras. Specifically, Humphreys defines B_l to be the algebra of $(2l+1) \times (2l+1)$ matrices (with complex entries) a so that $a^T \Omega + \Omega\, a = 0$, where

$$\Omega \;=\; \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & I_l \\ 0 & I_l & 0 \end{pmatrix}. \tag{1.23}$$

Of course this is not the definition of the orthogonal algebra that was given above, but B_l and $\mathfrak{o}(2l+1)\otimes\mathbb{C}$ are isomorphic by conjugacy. Finally the algebra D_l consists of the $2l \times 2l$ matrices (again with complex matrices) a so that $a^T\Omega + \Omega\, a = 0$, where now

$$\Omega \;=\; \begin{pmatrix} 0 & I_l \\ I_l & 0 \end{pmatrix}. \tag{1.24}$$

Again, the algebras D_l and $\mathfrak{o}(2l)\otimes\mathbb{C}$ are isomorphic via conjugacy.

Chapter 2

Fundamental definitions, and Engel's Theorem

September 11, 2012

2.1 Basic Definitions

A *representation* of a Lie algebra L is a homomorphism φ of L into the Lie algebra $\mathfrak{gl}(V)$ for some vector space V over $\mathbb{F}$. Every Lie algebra has at least one representation, the adjoint representation $ad : L \to End(V)$.

A *subalgebra* K of L is a subspace that is closed under the bracket.

An *ideal* I of K is a subalgebra so that $x \in L$, $y \in I$ implies $[x, y] \in I$.

If I and J are ideals, so is $I + J$, defined to be the set of all elements $c_1 x + c_2 y$ where $c_1, c_2 \in \mathbb{F}$, $x \in I$, $y \in J$.

If I and J are ideals, so is $[I,]\, J]$, which is defined to be the vector space spanned by elements of the form $[x,\, y]$ where $x \in J$ and $y \in J$.

Two ideals possessed by any Lie algebra are the *derived algebra* $[L,\, L]$, and the *center* $Z(L)$ or $C(L)$, defined to be the set of elements $x \in L$ so that $[x, y] = 0$ for all $y \in L$. Either of these algebras may be trivial or may equal L itself.

Proposition 2.1.1 (Humphreys 2.2) *Assuming $I, J \subset L$ are ideals. Then $(I + J)/J$ is canonically isomorphic to $I/(I \cap J)$, and if $I \subseteq J$ then K/J is canonically isomorphic to*

$(K/I)/(J/I)$

If $K \subseteq L$ is a subalgebra, we define the *normalizer* of K in L

$$N_L(K) \;=\; \{\, x \in L \,\big|\, [x, K] \in K \,\}. \tag{2.1}$$

Using the Jacobi identity, $N_L(K)$ can be seen to be a subalgebra of L. It is the largest subsalgebra that contains K as an ideal. If $N_L(K) = L$ then K is an ideal. If $N_L(K) = K$ then K is said to be self-normalizing.

If $K \subseteq L$ is a subalgebra, we define the *cetralizer* of K in L to be

$$C_L(K) \;=\; \{\, x \in L \,\big|\, [x, K] = 0 \,\}. \tag{2.2}$$

Note that K is usually not contained in $C_L(K)$. Also, $C_L(L) = C(L)$.

A *derivation* of a Lie algebgra L is a linear map $\delta : L \to L$ so that $\delta[x, y] = [\delta x, y] + [x, \delta y]$. The vector space of derivations $Der(L)$ is in fact a Lie algebra; it is easily checked that if $\delta, \delta' \in Der(L)$ then $[\delta, \delta'] \triangleq \delta\delta' - \delta'\delta$ is also a derivation. By the Jacobi identity, the adjoint map can be thought of as $ad : L \to Der(L)$.

If $[L, L] = \{0\}$ then L is called *abelian*. If L has no nontrivial proper ideals, then L is called *simple*.

If L, N are Lie algebras, a *homomorphism* $\varphi : L \to N$ is a linear map that commutes with the backets; that is

$$\varphi\left([x,\, y]\right) \;=\; [\varphi(x),\, \varphi(y)] \tag{2.3}$$

whenever $x, y \in L$. The *Kernel* of φ, denote $Ker(\varphi)$ is the vector space kernel of the map φ. It is easily checked that $Ker(\varphi)$ is an ideal.

A homomorphism $\varphi : L \to N$ is called a *monomorphism* if $Ker(\varphi)$ is the trivial subspace. It is called an epimorphism if its image is N. It is called an isomorphism if it is a monomorphism and an epimorphism.

Proposition 2.1.2 (Humphreys 2.2) *If $\varphi : L \to N$ is a homomorphism, then $Im(\varphi)$ is canonically ismorphic to $L/Ker(\varphi)$.*

A Lie algebra L is called a linear Lie algebra if it is a subalgebra of $\mathfrak{gl}(V)$ for some finite dimensional vector space V.

Proposition 2.1.3 *Any simple Lie algebra is isomorphic to a linear Lie algebra*

2.2 Solvable Lie algebras

Let L be a Lie algebra. We can define its derived series $L^{(0)}$, $L^{(1)}$, … by $L^{(0)} = L$ and

$$L^{(k)} = \left[L^{(k-1)}, L^{(k-1)}\right]. \tag{2.4}$$

We call L *solvable* if $L^{(n)} = \{0\}$ for some n.

Proposition 2.2.1 (Humphreys 3.1) *Let L be a Lie algebra.*

a) If L is solvable, so are all subalgebras and all homomorphic images.

b) If $I \subseteq L$ is a solvable ideal and L/I is solvable, then L is solvable.

c) If $I, J \subseteq L$ are solvable ideals, then $I + J$ is a solvable ideal.

Pf. Easy. □

2.3 Nilpotent Lie algebras

Let L be a Lie algebra. We can define its descending central series L^0, L^1, … by $L^0 = L$ and

$$L^k = \left[L, L^{k-1}\right]. \tag{2.5}$$

We call L *nilpotent* if $L^n = \{0\}$ for some n.

Proposition 2.3.1 (Humphreys 3.2) *Let L be a Lie algebra.*

a) If L is nilpotent, so are all subalgebras and all homomorphic images.

b) If $L/Z(L)$ is nilpotent, so is L.

c) If L is nilpotent, then $Z(L)$ is not trivial.

Pf. a) Easy
b) If $L/Z(L)$ is nilpotent then $L^n \subseteq Z(L)$ for some L. Then $L^{n+1} = \{0\}$.
c) There is some n so that $L^n = \{0\}$ but $L^{n-1} \neq \{0\}$. Clearly $\{0\} = L^n = [L, L^{n-1}]$ implies $L^{n-1} \subseteq Z(L)$. □

Definition An element $x \in L$ is called *ad-nilpotent* if $(adx)^n = 0$ for some n. A Lie algebra L is called *ad-nilpotent* if every element of L is ad-nilpotent.

If $x, y \in L$, then $(adx)^n y = [x, [x, \dots, [x, y] \dots],] \in L^{n+1}$. Thus if L is nilpotent, it is ad-nilpotent.

Theorem 2.3.2 (Engel's Theorem) *If L is ad-nilpotent, it is nilpotent*

Theorem 2.3.3 *If L is a subalgebra of $\mathfrak{gl}(V)$ (V finite dimensional) and every $x \in L$ is a nilpotent transformation (meaning given x there is some $n \in \mathbb{N}$ so that $x^n.v = 0$ whenever $v \in V$), then there is some $v \in V$ so that $x.v = 0$ for all $x \in L$.*

Pf. Induction on the dimension of L. The theorem is clearly true for all L with $dim(L) = 1$. This is because $x \in L$ implies $x^n.v = 0$ for some n, so that there is a largest $i \in \ltimes$ with $x^i.v \neq 0$ but $x^{i+1}.v = 0$, in which case $v_i = x^i.v$ is a zero eigenvector.

Assume the theorem is true for all Lie algebras K with $dim(K) < dim(L)$. Let K be any maximal subalgebra of L—clearly subalgebras exist, for instance 1-dimensional subalgebras. We will prove first that K has codimension 1, and since the theorem is true for the action of K on V we are left just a single basis element whose action must be checked.

To prove K has codimension 1, consider the adjoint action of K on L/K (of course L/K is not a Lie algebra but only a vector space; still the action of K is well-defined (check)). By the inductive hypothesis, there is some vector $z \in L$ so that $z + K \in L/K$ is a zero-eigenvector for every element of K. This means that $K + \mathbb{F}z$ is also a subalgebra that strictly contains K, implying that either K was not maximal (which it is) or that $K + \mathbb{F}z$ is in fact L, verifying that K has codimension 1. Since $L = K + \mathbb{F}z$ and we showed$[z, k] \in K$ when $k \in K$, K is an ideal.

Now let $W \subseteq V$ be the subspace consisting of all zero-eigenvectors for K, or

$$W \ = \ \{\, w \in V \mid k.w = 0 \; when \; k \in K \,\}. \tag{2.6}$$

To see that L fixes W, let $w \in W$, $y \in L$, and $k \in K$. Since

$$k.y.w \ = \ y.k.w \ + \ [k,\, y].w \tag{2.7}$$

Because $k, [k, y] \in K$, the right-side is zero. Therefore $y.w \in W$. Using again $L = K + \mathbb{F}z$ and knowing that z has a nilpotent action on V and therefore on W, there must be a zero-eigenvector w' for z in W. Thus w' is a zero-eigenvector for all of L. □

Proof of Engel's theorem. Again we argue inductively on the dimension of L, assuming Engel's theorem holds for all Lie algebras K with $dim(K) < dim(L)$. The adjoint action $ad : L \to \mathfrak{gl}(L)$ expresses L as an algebra of nilpotent endomorphisms, which therefore have a common zero-eigenvector. Thus $Z(L)$ is non-trivial. Since $L/Z(L)$ is ad-nilpotent, and therefore nilpotent by the induction hypothesis, L is also nilpotent. □

Chapter 3

Lie's Theorem

September 13, 2012

3.1 Weights and Weight Spaces

Proposition 3.1.1 *If L is a vector space of endomorphisms of V and $v \in V$ is an eigenvector common to all endomorphisms $x \in L$, then v determines a linear operator $\lambda_v : L \to \mathbb{F}$, defined implicitly by*

$$x.v \;=\; \lambda_v(x)\, v. \tag{3.1}$$

Pf. To verify linearity, we have

$$(c_1x \,+\, c_2y).v \;=\; \lambda_v(c_1x \,+\, c_2y)\, v \tag{3.2}$$

and

$$(c_1x \,+\, c_2y).v \;=\; c_1x.v \,+\, c_2y.v \;=\; c_1\lambda_v(x)\, v \,+\, c_2\lambda_v(y)\, v. \tag{3.3}$$

so $\lambda_v(c_1x \,+\, c_2y) = c_1\lambda_v(x) \,+\, c_2\lambda_v(y)$. □

If L is a Lie algebra with a representation $\varphi : L \to \mathfrak{gl}(V)$ and if $\alpha : L \to \mathbb{F}$ is a linear functional, we define

$$V_\alpha \;=\; \{\, v \in V \mid x.v = \alpha(x)v \;\text{ for all }\; x \in L \,\}. \tag{3.4}$$

If V_α is non-trivial, we call the functional α a *weight* for the representation, and V_α the *weight space* corresponding to α.

3.2 Lie's Theorem

Unlike Engel's theorem which required no special restriction on the base field, Lie's theorem requires both algebraic closure and characteristic 0, so realistically, although there are other possibilities, we are working with $\mathbb{F} = \mathbb{C}$.

Theorem 3.2.1 *Let L be a solvable subalgebra of $\mathfrak{gl}(V)$, where V is finite dimensional and non-trivial. Assume also that $\mathbb{F}$ is algebraically closed and $char(\mathbb{F}) = 0$. Then V contains an eigenvector common to all endomorphisms in L.*

Pf. To imitate the proof of Engel's theorem, we use induction on the dimension of L, and assume the theorem holds for all algebras K with $dim(K) < dim(L)$.

Step 1: Identifying a codimension 1 ideal. We first find an ideal of codimension 1 in L. Note that $[L, L]$ is a strictly proper ideal; if $[L, L] = \{0\}$ then L is abelian, hence nilpotent, and by Engel's theorem we are done. Since $L/[L, L]$ is abelian, and codimension 1 subspace of $L/[L, L]$ is an ideal, and taking the inverse image along the projection $L \to L/[L, L]$ we have a codimension 1 ideal in L. Call it K.

Step 2: Identifying a weight space for K. There is at least one common eigenvector $w \in V$ of K, which induces a linear operator $\lambda_w : K \to \mathbb{F}$. Let $W_w \subseteq V$ be the subspace of common eigenvectors that induce the same linear operator on K. That is

$$W_w \;=\; \{\, u \in V \mid \lambda_u = \lambda_w \,\} \tag{3.5}$$

Next we show that $L = K + \mathbb{F}z$ leaves W_w invariant. To see this, let $u \in W_w$; we must show that for $k \in K$ and $l \in L$ we have $k.l.u = \lambda_w(k)\, l.u$. We have

$$k.l.u \;=\; l.k.u + [k, l].u \;=\; \lambda_w(k)\, l.u + \lambda_w([k, l])\, u \tag{3.6}$$

so we must show that $\lambda_w([k, l]) = 0$.

Step 3: Proving L fixes the weight space. Consider the space $U_n \subseteq V$ spanned by an arbitrary $u \in W_w$ and powers of z acting on u up to the $(n-1)^th$ power:

$$U_n \;=\; span_{\mathbb{F}}\,\{u,\, z.u,\, \dots,\, z^n.u\,\}\,. \tag{3.7}$$

This gives a sequence of nested subspaces

$$\mathbb{F}u \;=\; U_1 \subseteq U_2 \subseteq U_3 \subseteq \dots \tag{3.8}$$

which obviously terminates. We show that K stabilizes each U_n. Letting $k \in K$, then on $U_1 = \mathbb{F}u$ we have

$$k.u \;=\; \lambda_w(k)u \;\in\; U_1. \tag{3.9}$$

On $U_2 = \mathbb{F}u \oplus \mathbb{F}z.u$ we have

$$k.z.u \;=\; z.k.u \,+\, [k,z].u \;=\; \lambda_w(k)\, z.u \,+\, \lambda_w([k,z])\, u \tag{3.10}$$

which is in U_2. Continuing inductively, we have that K stabilizes each U_n. Further, from this calulation and an induction, it is clear that, on the natural basis for U_n we have

$$k \;=\; \begin{pmatrix} \lambda_w(k) & \lambda_w([k,z]) & \lambda_w([[k,z],z]) & \dots & \lambda_w([\dots[k,z],\dots,z]) \\ 0 & \lambda_w(k) & 2\lambda_w([k,z]) & & \\ 0 & 0 & \lambda_w(k) & & \\ 0 & 0 & 0 & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \lambda_w(k) \end{pmatrix} \tag{3.11}$$

in other words k can be expressed as an upper-triangular matrix, and

$$n\lambda_w(k) \;=\; Tr_{U_n}(k). \tag{3.12}$$

Taking U_n to be *maximal*, we have that k and z both act as endomorphisms on U_n, and so $[k,z] \in \mathfrak{sl}(U_n)$. Therefore $Tr_{U_n}([k,z]) = 0$, and so, because $\mathbb{F}$ has characteristic 0, we have $\lambda_w([k,z]) = 0$, as desired.

Thus $L = K + \mathbb{F}z$ stabilizes W_w.

Step 4: Finding a common eigenvector for L. Since $\mathbb{F}$ is algebraically closed and z fixes W_w, it has a complete set of eigenvectors on W_w. Any such eigenvector is therefore an eigenvector common to K and z, so is an eigenvector common to L. □

Corollary 3.2.2 (Lie's Theorem) *Let L be a solvable subalgebra of $\mathfrak{gl}(V)$ for some finite dimensional V. Assume further that the base field is compelete and has characterstic 0. Then there exists a flag*

$$\{0\} \;=\; V_0 \;\subset V_1 \;\subset\; \dots \;\subset\; V_n \;=\; V \tag{3.13}$$

that is stabilized by L. Further, it is possible to take each V_i to be of codimension 1 in V_{i-1}.

Pf. The theorem along with induction. Letting v_1 be a common eigenvector and setting $V_1 = \mathbb{F}x$, the quotient space V/V_1 has a well-defined action by L (an easy check). Then V/V_1 has an eigenvector common to L, leading to an element $v_2 \in V$ whose image under any $l \in L$ is a multiple of v_2 plus an element of V_1. Continuing inductively, the theorem follows. □

Corollary 3.2.3 *If L is a solvable, finite-dimensional Lie algebra over a complete field of characteristic 0, then L has a chain of ideals*

$$\{0\} \;=\; L_0 \;\subset\; L_1 \;\subset\; \dots \;\subset\; L_n \;=\; L \tag{3.14}$$

each of codimension 1 in the next.

Pf. Apply Lie's theorem to $ad\,L$. □

Corollary 3.2.4 *If L is solvable, then any element of $[L,L]$ is ad-nilpotent in L, and $[L,L]$ is a nilpotent algebra.*

Pf. The ad-action of each $x, y \in L$ can be expressed as an upper-triangular matrix

$$ad_L x = \begin{pmatrix} a_1 & & & \\ 0 & a_2 & & \\ \vdots & & \ddots & \\ 0 & 0 & \dots & a_n \end{pmatrix} \tag{3.15}$$

$$ad_L y = \begin{pmatrix} b_1 & & & \\ 0 & b_2 & & \\ \vdots & & \ddots & \\ 0 & 0 & \dots & b_n \end{pmatrix}. \tag{3.16}$$

Thus $ad_L x\, ad_L y$ is an upper-triangular matrix

$$\begin{pmatrix} a_1 b_1 & & & \\ 0 & a_2 b_2 & & \\ \vdots & & \ddots & \\ 0 & 0 & \dots & a_n b_n \end{pmatrix}. \tag{3.17}$$

Therefore

$$ad_L[x,y] = ad_L x\, ad_L y - ad_L y\, ad_L x \tag{3.18}$$

is strictly upper-triangular, and therefore a nilpotent operator on L. Since $ad_L x$ stabilizes $[L,L]$, $ad_L[L,L]\big|_{[L,L]} = ad_{[L,L]}[L,L]$, so that any element of $[L,L]$ is ad-nilpotent with respect to the algebra $[L,L]$. Engel's theorem now gives the nilpotency of $[L,L]$. □

Chapter 4

The Fitting and Jordan-Chevalley decompositions

September 18, 2012

4.1 Minimal and Characteristic polynomials

If $x \in End(V)$, its *minimal polynomial* M is the monic polynomial of smallest degree so that the transformation $M(x) \in End(V)$ is zero. The *characteristic polynomial* Ch is a polynomial in one variable given by

$$Ch(\lambda) \;=\; \det(\lambda I \,-\, x). \tag{4.1}$$

Clearly it is monic. Plugging x itself into the resulting polynomial and formally evaluating, we clearly have that $Ch(x)$ is the zero endomorphism. We have

$$M(\lambda) \;=\; \prod_{i=1}^{n} (\lambda \,-\, \lambda_i)^{e_i} \tag{4.2}$$

$$Ch(\lambda) \;=\; \prod_{i=1}^{n} (\lambda \,-\, \lambda_i)^{f_i} \tag{4.3}$$

$$\tag{4.4}$$

and that $f_i \geq e_i$. The numbers λ_i are the eigenvalues of x, f_i is the multiplicity of λ_i, and e_i is the degree of λ_i.

4.2 The Fitting Decomposition

Let x be a linear operator on a finite dimensional vector space V. Then V is isomophic to a direct sum $V = V_{0,x} \oplus V_{1,x}$, where $V_{i,x}$ is a subspace of V. Further, $V_{0,x}$ has the property that $x.V_{0,x} \subset V_{0,x}$ and for some $n \in \mathbb{N}$ we have

$$x^n.V_{0,x} \;=\; \{0\}, \tag{4.5}$$

and $V_{1,x}$ has the property that

$$x.V_{1,x} \;=\; V_{1,x} \tag{4.6}$$

or that x is an isomorphism on $V_{1,x}$. With respect to x, $V_{0,x}$ is called the *Fitting null space* and $V_{1,x}$ is called the *Fitting one space*.

First, define

$$V_{0,x} \;=\; \{\, v \in V \mid x^i.v \text{ for some } i \in \mathbb{N}\,\}. \tag{4.7}$$

Since

$$V \;\supseteq\; x.V \;\supseteq\; x^2.V \;\supseteq\; \ldots \tag{4.8}$$

and V is finite dimensional, there is some r so that $x^r.V = x^{r+1}.V = \ldots$. Simply define $V_{1,x}$ to be this $x^r.V$.

Clearly $V_{0,x}$ and $V_{1,x}$ are invariant under x. Now let $w \in V$; we show that $w \in V_{0,x} \oplus V_{1,x}$. Letting $r \in \mathbb{N}$ be large enough that $x^r.V_{0,x} = \{0\}$, we see that $x^r.V = V_{1,x}$ implies

$$x^r.w \;\in\; V_{1,x}. \tag{4.9}$$

But x, hence x^r is an isomorphism on $V_{1,x}$, so there is some $w_1 \in V_{1,x}$ with $x^r.w = x^r.w_1$. But then $x^r.(w - w_1) = 0$, so setting $w_0 = w - w_1$ we have

$$w \;=\; w_0 \;+\; w_1. \tag{4.10}$$

A further decomposition exists. If μ is any polynomial in one variable, set

$$V_\mu \;=\; \{\, v \in V \mid \mu(x)^r.V = 0 \text{ for some } r \in \mathbb{N}\,\}. \tag{4.11}$$

Let π_i be the irreducible factors of the minimal polynomial of x, so $M(\lambda) = \prod_{i=1}^{n} \pi_i(\lambda)^{e_i}$, and consider the spaces V_{π_i}. Since x commutes with $\pi_i(x)$, we have $x.V_{\pi_i} \subseteq V_{\pi_i}$.

Proposition 4.2.1 *If $\pi_i(\lambda) \neq \lambda$, then the restriction of x to V_{π_i} is an isomorphism. Futher, it is the sum of two operators $x_{s,i}$ and $x_{n,i}$, with the following properties. The operator $x_{s,i}$ acts on V_{π_i} by constant multiplication, and $x_{n,i} : V_{\pi_i} \to V_{\pi_i}$ is a nilpotent endomorphism. Finally, $x_{s,i}$ and $x_{n,i}$ are the unique operators on V_{π_i} with these properties.*

Pf. Were $x : V_{\pi_i} \to V_{\pi_i}$ to have a zero eigenvector v_0, then

$$\begin{aligned} 0 \;=\; \pi_i(x)^{e_i}.v_0 &= (x - \lambda_i I)^{e_i - 1}.(x - \lambda_i I).v_0 && (4.12)\\ &= \lambda_i (x - \lambda_i I)^{e_i - 1}.v_0 && (4.13)\\ &\vdots && (4.14)\\ &= \lambda_i^{e_i} v_0 && (4.15) \end{aligned}$$

so $\lambda_i = 0$, which is false by hypothesis. Thus $x : V_{\pi_i} \to V_{\pi_i}$ is an isomorphism.

Of course x is the sum of the operators $x_{s,i} = \lambda_i I$ and $x_{n,i} = x - \lambda_i I$, and $x - \lambda_i I$ is nilpotent on V_{π_i} by definition. The three operators x, $\lambda_i I$, and $x - \lambda_i I$ clearly commute.

If $x'_{s,i}$ is another operator on V_{π_i} that acts by constant multiplication and $x - x'_{s,i}$ is nilpotent, then $x'_{s,i} - x_{s,i}$ acts by constant multiplication. Clearly also $x'_{s,i}$ and $x_{s,i}$ commute; an easy computation then shows $x'_{n,i}$ and $x_{n,i}$ commute as well. Also, $x'_{n,i} - x_{n,i} = (x - x'_{s,i}) - (x - x_{s,i}) = x_{s,i} - x'_{s,i}$. Since $x'_{n,i} - x_{n,i}$ is nilpotent, so is $x_{s,i} - x'_{s,i}$, which is impossible unless it is zero. □

Proposition 4.2.2 *Let $M(\lambda) = \prod_{i=0}^{n} \pi_i(\lambda)^{e_i}$ be the minimal polynomial for the endomorphism $x : V \to V$, where $\pi_0(\lambda) = \lambda$ (and possibly $e_0 = 0$). The vector space $V_{1,x}$ has the decomposition*

$$V_{1,x} \;=\; \bigoplus_{i=1}^{n} V_{\pi_i} \tag{4.16}$$

Therefore the vector space V has the decomposition

$$V \;=\; \bigoplus_{i=0}^{n} V_{\pi_i}. \tag{4.17}$$

Pf. We have $V_{0,x} = V_{\pi_0}$ by definition. Since x is an isomorphism on each V_{π_i} when $i \neq 0$, we have

$$x^r.\left(\bigoplus_{i=1}^{n} V_{\pi_i}\right) \;=\; \bigoplus_{i=1}^{n} V_{\pi_i} \tag{4.18}$$

so that $\bigoplus_{i=1}^{n} V_{\pi_i} \subseteq V_{1,x}$. If there is some $v \in V_{1,x} \setminus \bigoplus_{i=1}^{n} V_{\pi_i}$, then

$$W \;=\; V_{1,x} / \bigoplus_{i=1}^{n} V_{\pi_i} \tag{4.19}$$

is non-trivial, and is acted on by x. Further, because the action of x on $V_{1,x}$ is by isomorphism, its action on W is also an isomorphism. Therefore there is some $w \in W$ with

$x.w = cw$ for some non-zero $c \in \mathbb{F}$. Letting $v' \in V_{1,x}$ be in the inverse image of w under the quotient $V_{1,x} \to W$, we have

$$x.v' \;=\; cv' \,+\, v_1 \tag{4.20}$$
$$x^{e_0}.v' \;=\; c^{e_0}v' \,+\, v_2 \tag{4.21}$$

where $v_1, v_2 \in \bigoplus_{i=1}^n V_{\pi_i}$ Now let $Q(\lambda) = \prod_{i=1}^n \pi_i(\lambda)^{e_i}$. We have that $Q(x)$ kills $\bigoplus_{i=1}^n V_{\pi_i}$, $x^{e_0}Q(x)$ kills V, and $Q(x)$ commutes with x. Thus

$$x^{e_0}.v' \;=\; c^{e_0}v' \,+\, v_2 \tag{4.22}$$
$$x^{e_0}.Q(x).v' \;=\; c^{e_0}Q(x).v'. \tag{4.23}$$

Since c is non-zero, this forces $Q(x)$ to kill v', which means $v' \in \bigoplus_{i=1}^n V_{\pi_i}$.

□

Corollary 4.2.3 (The Jordan-Chevalley decomposition) *If x is an endomorphism on V, then $x = x_s + x_n$, where x_s and x_n have the following properties: the roots of the minimal polynomial of x_s are distinct, x_n is nilpotent, and x_s and x_n commute. Further, x_s and x_n are the unique operators that satisfy these properties.*

Pf. By Proposition (4.2.1), we have $x|_{V_{\pi_i}} = x_{s,i} + x_{n,i}$, where all three operators commute. We can extend the action of $x_{s,i}, x_{n,i}$ from V_{π_i} to V by requiring them to act as multiplication by zero on all V_{π_j} when $i \neq j$. This makes sense because $V = \bigoplus_{i=0}^n V_{\pi_i}$, so we have well-defined projections onto the V_{π_i}. Further, all the $x_{s,i}, x_{n,i}$ operators commute. Defining

$$x_s \;=\; \sum_{i=0}^n x_{s,i} \qquad \text{and} \qquad x_n \;=\; \sum_{i=0}^n x_{n,i} \tag{4.24}$$

provides the required operators. Now assume x'_s and x'_n are other choices for the semi-simple and nilpotent parts of x. But then $x_s - x'_s = x_n - x'_n$. Because $x - x'_s$ is nilpotent, x and x'_s have the same eigenspace decomposition, so x'_s acts by constant multiplication on each V_{π_i}. Clearly then $x_s = x'_s$, so also $x_n = x'_n$. □

If $x : V \to V$ is an endomorphism, x_s is called its *semisimple* part and x_n is called its *nilpotent* part.

Proposition 4.2.4 *Given an endomorphism $x : V \to V$, where V is a finite dimensional vector space over the algebraically complete field $\mathbb{F}$, there are polynomials $P_s(\lambda)$, $P_n(\lambda)$ without constant terms so that $x_s = P_s(x)$ and $x_n = P_n(x)$. Further, x, x_s, and x_n commute, and if x'_s, x'_n are other semi-simple and nilpotent endomorphisms with $x = x'_s + x'_n$, then $x'_s = x_s$ and $x'_n = x_n$.*

Pf. As usual let

$$M(\lambda) \;=\; \prod_{i=0}^n (\lambda \,-\, \lambda_i)^{e_i} \tag{4.25}$$

be the minimal polynomial of x. Consider the (commutative) algebra in $End(V)$ generated by x and the identity I. By the Chinese remainder theorem, we can find an element P_s of this algebra that satisfies the following congruences:

$$P_s(x) \equiv 0 \quad (mod\; x) \tag{4.26}$$
$$P_s(x) \equiv \lambda_1 I \quad (mod\; (x - \lambda_1 I)^{e_1}) \tag{4.27}$$
$$\vdots \tag{4.28}$$
$$P_s(x) \equiv \lambda_n I \quad (mod\; (x - \lambda_n I)^{e_n}) \tag{4.29}$$

Set $P_n(x) = x - P_s(x)$. Since $P_s(x) \equiv 0\,(mod\, x)$ we have that x is a factor of $P_s(x)$, so neither P_s nor P_n have a constant term.

These congruences mean in particular that $P_s(x) - \lambda_i I = q_i(x) \cdot (x - \lambda_i I)^{e_i}$ for some polynomials $q_i(x)$. Since $(x - \lambda_i I)^{e_i}$ acts by 0 on V_{π_i}, we have that $P_s(x)$ acts as constant multiplication by λ_i on V_{π_i}. Thus $P_s(x)$ is semisimple. Define $x_s = P_s(x)$ and $x_n = P_n(x)$. Clearly x_s commutes with x. Further, it is clear that the only eigenvalue of $x_n = x - x_s$ is zero, so x_n is nilpotent.

If x_s' were another semisimple endomorphism with $x - x_s'$ nilpotent, then since $x - x_s'$ has all zero eigenvalues, it must be the case that x_s' and x_s have the same action on each V_{π_i}. Thus they are the same. □

It should be noted that x acts on L via the adjoint representation. Sometimes this allows x to be separated into its *abstract* semisimple and nilpotent parts. Clearly $ad\,x \in End(L)$ has components $ad\,x = (ad\,x)_s + (ad\,x)_n$, but when does $(ad\,x)_s = ad\,x_s$, $(ad\,x)_n = ad\,x_n$ for elements $x_s, x_n \in L$? The most we will prove today is the following.

Proposition 4.2.5 *Assume $x \in End(V)$ has Jordan decomposition $x = x_s + x_n$. Then $ad\,x \in End(End(V))$ has Jordan decomposition $ad\,x = ad\,x_s + ad\,x_n$.*

Pf. An inductive argument easily shows that $ad\,x_n$ is nilpotent if x_n is; indeed if $N \in \mathbb{N}$ has $(x_n)^N = 0$ then $(ad\,x_n)^{2N} = 0$. To see that $ad\,x_s$ is semi-simple when x_s is, pick a basis $\{v_1, \ldots, v_n\}$ so that $x_s.v_i = \lambda_i\, v_i$. Then $V = span\{v_1, \ldots, v_n\}$ is identified with $\mathbb{C}^n$, so we can let the usual matrices $\{e_{ij}\}_{i,j=1}^n$ be the basis for $\mathfrak{gl}(V)$. We have $e_{ij}(v_k) = \delta_{jk} v_i$, and

$$\begin{aligned} (ad\,x_s\,(e_{ij})\,)v_k &= x_s.e_{ij}.v_k - e_{ij}.x_s.v_k \\ &= \delta_{jk} x_s.v_i - \lambda_k\, e_{ij}.v_k \\ &= (\lambda_i - \lambda_k)\;\delta_{jk}\, v_i \\ &= (\lambda_i - \lambda_j)\;\delta_{jk}\, v_i \;=\; (\lambda_i - \lambda_j)\; e_{ij}.v_k \end{aligned} \tag{4.30}$$

so $(ad\,x_s)(e_{ij}) = (\lambda_i - \lambda_j) e_{ij}$. Thus $ad\,x_s$ acts diagonally on $\mathfrak{gl}(V)$ and is therefore semisimple. By the uniqueness of the Jordan decomposition therefore $(ad\,x)_s = ad\,x_s$ and $(ad\,x)_n = ad\,x_n$. □

Chapter 5

Cartan's criterion and semisimplicity

September 20, 2012

5.1 Cartan's Criterion

Theorem 5.1.1 *Assume the base field is $\mathbb{C}$, and let $A \subseteq B$ be vector subspaces of $End(V)$. Set $M = \{x \in End(V) \,|\, [x, B] \subseteq A\}$. If $x \in M$ and $Tr(xy) = 0$ for any other $y \in M$, then x is nilpotent.*

Pf. Choose a basis $\{v_1, \ldots, v_a, v_{a+1}, \ldots, v_b, v_{b+1}, \ldots, v_n\}$ for V so that $\{v_1, \ldots, v_a\}$ is a basis for A and $\{v_1, \ldots, v_b\}$ is a basis for B. Let $\bar{x} \in End(V)$ be the operator whose matrix is given by the complex conjugate of the matrix for x. Clearly $\bar{x} \in M$.

Let P_s be the polynomial without constant term so that

$$(ad\, x)_s \;=\; P_s(ad\, x) \tag{5.1}$$

Because $ad\, x : B \to A$, we have also $(ad\, x)_s : B \to A$, due to the fact that P_s has no constant term. But also $x = x_s + x_n$, and by a previous lemma, $ad\, x_s = (ad\, x)_s$. Because $ad\, x_s = (ad\, x)_s : B \to A$, we have $x_s \in M$.

It is easily verified that the polynomial producing $(ad\, \bar{x})_s$ is just the polynomial obtained from P_s by taking the complex conjugates of all coefficients. Thus, as matrices,

$\bar{x}_s = \overline{x_s}$. Thus

$$\begin{aligned} 0 &= Tr(x\,\bar{x}) \quad (5.2)\\ &= Tr\left((x_s + x_n)(\bar{x}_s + \bar{x}_n)\right) \quad (5.3) \end{aligned}$$

by since x_s, $\bar{x}_n$ commute, $x_s\bar{x}_n$ is nilpotent and therefore of trace 0, and likewise for $\bar{x}_s x_n$. Thus

$$0 \;=\; Tr(x\,\bar{x}) \;=\; Tr(x_s\bar{x}_s) \;=\; |\lambda_1|^2 \,+\, \ldots \,+\, |\lambda_n|^2 \quad (5.4)$$

so that $\lambda_i = 0$ for all i. Therefore x is nilpotent. □

Note that for any operators $x, y \in \mathfrak{gl}(V)$, we have $Tr([x, y]) = 0$.

Corollary 5.1.2 (Cartan's Criterion) *Let* $\mathfrak{g}$ *be a Lie algebra over* $\mathbb{C}$*. If* $x \in [\mathfrak{g}, \mathfrak{g}]$ *implies* $Tr(ad\,x\;ad\,y) = 0$ *for all* $y \in \mathfrak{g}$*, then* $\mathfrak{g}$ *is solvable.*

Pf. In the context of Theorem 5.1.1, let $A = ad\,[\mathfrak{g}, \mathfrak{g}]$ and $B = ad\,\mathfrak{g}$, both subsets of $\mathfrak{gl}(\mathfrak{g})$. Clearly M contains $B = ad\,\mathfrak{g}$, but there may be other elements in M. Let $Z \in M \setminus ad\,\mathfrak{g}$; we must show that still $Tr(Z\,ad\,x) = 0$.

Since $Z \in M$ we have, by definition, $[Z, ad\,a] \in ad\,[\mathfrak{g}, \mathfrak{g}] = A$ whenever $a \in \mathfrak{g}$. Consider a typical generator $ad\,[a, b]$ of $ad\,[\mathfrak{g}, \mathfrak{g}]$; we have to show $Tr(Z\,ad[a, b]) = 0$. We ahve

$$Tr(Z\,ad\,[a, b]) \;=\; Tr(Z\,[ad\,a,\; ad\,b]) \quad (5.5)$$

so $Z[ad\,a,\; ad\,b] - [Z,\; ad\,a]\,ad\,b \;=\; [ad\,a,\; Z\,ad\,b]$ and $Tr(Z\,[ad\,a,\; ad\,b]) \;=\; Tr([Z,\; ad\,a]\,ad\,b)$. Thus

$$Tr(Z\,ad\,[a, b]) \;=\; Tr([Z,\; ad\,a]\,ad\,b) \quad (5.6)$$

But since $[Z,\; ad\,a] \in ad\,[\mathfrak{g}, \mathfrak{g}]$, the right side is zero by hypothesis.

Therefore $Tr(Z\,ad\,x) \;=\; 0$ whenever $x \in [\mathfrak{g}, \mathfrak{g}]$ and $Z \in M$. By Theorem 5.1.1 we conclude that any such $ad\,x$ is a nilpotent operator. Since $[\mathfrak{g}, \mathfrak{g}]$ is therefore ad-nilpotent, by Engel's theorem it is nilpotent. Hence $\mathfrak{g}$ is solvable. □

5.2 The Killing Form

5.2.1 Basic concepts

The bilinear map $Tr(ad\,x\,ad\,y)$ for $x, y \in \mathfrak{g}$ and $ad : \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$ is used so often that it is given a name, the *Killing form*. Specifically, we define

$$\kappa(x,\; y) \;=\; Tr\,(ad\,x\,ad\,y). \quad (5.7)$$

To be clear about context, we often write $\kappa_{\mathfrak{g}}$ to indicate what Lie algebra the adjoint map is acting on. For instance, if $\mathfrak{l} \subset \mathfrak{g}$ is a proper subalgebra, it is entirely possible that $\kappa_{\mathfrak{l}} \neq \kappa_{\mathfrak{g}}|_{\mathfrak{l}}$.

The *null-space*, or *radical*, of $\kappa_{\mathfrak{g}}$ is the subspace $Rad\,\kappa_{\mathfrak{g}} \subseteq \mathfrak{g}$ of elements $x \in \mathfrak{g}$ so that $\kappa_{\mathfrak{g}}(x, \cdot) = 0$.

If $W \subseteq \mathfrak{g}$ is any vector subspace, we define

$$W^{\perp} \;=\; \left\{ x \in \mathfrak{g} \;\middle|\; \kappa_{\mathfrak{g}}(x,\, w) \;=\; 0 \text{ for all } w \in W \right\}. \tag{5.8}$$

It is entirely possible that $W^{\perp} \cap W$ is non-trivial. For instance if $\mathfrak{g}$ is commutative then $\mathfrak{g}^{\perp} = \mathfrak{g}$. Of course $Rad\,\kappa_{\mathfrak{g}} = \mathfrak{g}^{\perp}$.

5.2.2 The Heisenberg algebra $\mathfrak{h}$

This algebra is spanned by X, Y, H with $[X, Y] = H$, $[X, H] = 0$, and $[Y, H] = 0$ Thus as matrices we can express $ad\,X$, $ad\,Y$, and $ad\,H$ as

$$ad\,X \;=\; \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \tag{5.9}$$

$$ad\,Y \;=\; \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix} \tag{5.10}$$

$$ad\,H \;=\; \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{5.11}$$

Thus the matrix for $\kappa_{\mathfrak{h}}$ is

$$\kappa_{\mathfrak{h}} \;=\; \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{5.12}$$

so the Killing form is completely degenerate. This is necessarily the case, because the nilpotency of $\mathfrak{h}$ implies the nilpotency of each element in ad $\mathfrak{h}$, so each such element has only zero eigenvectors.

5.2.3 The algebra of upper triangular 2×2 matrices

This is the solvable algebra given by $\mathfrak{g} = span_{\mathbb{C}}\{X,\, Y,\, H\}$ with

$$[X,\, Y] \;=\; 0, \qquad [X,\, H] \;=\; H, \quad \text{and} \quad [Y,\, H] \;=\; -H. \tag{5.13}$$

Powers of $ad\,X$ or $ad\,Y$ never vanish so in particular $\mathfrak{g}$ is not nilpotent. We compute

$$ad\,X = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{5.14}$$

$$ad\,Y = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} \tag{5.15}$$

$$ad\,H = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -1 & 1 & 0 \end{pmatrix} \tag{5.16}$$

Thus

$$\kappa_{\mathfrak{g}} = \begin{pmatrix} 1 & -1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{5.17}$$

which is degenerate, and has rank 1. As we shall see, the degeneracy of $\kappa_{\mathfrak{g}}$ is related to the fact that $\mathfrak{g}$ is solvable.

5.2.4 The algebra $\mathfrak{sl}(2)$

This is the Lie algebra spanned by A, B, and H where

$$[A, B] = H, \qquad [H, A] = 2A, \quad \text{and} \quad [H, B] = -2B. \tag{5.18}$$

This algebra is simple. We have

$$ad\,A = \begin{pmatrix} 0 & 0 & -2 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \tag{5.19}$$

$$ad\,B = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 2 \\ -1 & 0 & 0 \end{pmatrix} \tag{5.20}$$

$$ad\,H = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{5.21}$$

Therefore

$$\kappa_{\mathfrak{sl}(2)} = \begin{pmatrix} 0 & 4 & 0 \\ 4 & 0 & 0 \\ 0 & 0 & 8 \end{pmatrix}. \tag{5.22}$$

The Killing form is non-singular with indefinite signature $(+, +, -)$. From Cartan's criterion we know that the definiteness of $\kappa_{\mathfrak{sl}(2)}$ is implied by its semi-simplicity. The indefinite signature is related to the fact that the (real) Lie group $SL(2, \mathbb{R})$ is non-compact.

5.2.5 The algebra $\mathfrak{o}(3)$

This is the Lie algebra spanned by X, Y, and H where

$$[X, Y] = Z, \qquad [Z, X] = Y, \quad \text{and} \quad [Y, Z] = X. \tag{5.23}$$

This algebra is simple. We have

$$ad\,X = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} \tag{5.24}$$

$$ad\,Y = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix} \tag{5.25}$$

$$ad\,Z = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{5.26}$$

Anyone familiar with 3-dimensional geometry knows these are the infinitesimal generators of the rotations about the X-, Y-, and Z-axes, respectively. That is, the operator $e^{\theta\, ad\, X} = Ad\, e^{\theta X} : \mathfrak{o}(3) \to \mathfrak{o}(3)$ is the rotation through the angle θ about the X-axis in $\mathfrak{o}(3) \approx \mathbb{R}^3$.

We compute

$$\kappa_{\mathfrak{o}(3)} = \begin{pmatrix} -2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{pmatrix}. \tag{5.27}$$

The Killing form is negative definite. The definiteness of the signature is related to the fact that the (real) Lie group $SO(3)$ is compact.

However, one might notice that $\mathfrak{o}(3, \mathbb{C})$ and $\mathfrak{sl}(2, \mathbb{C})$ are the same Lie algebra, a fact of some importance in quantum mechanics.

5.2.6 Applications

A rephrasing of Cartan's criterion is

Theorem 5.2.1 *If $\mathfrak{g}$ is a Lie algebra with the usual conditions*[1]*, and if $[\mathfrak{g}, \mathfrak{g}] \subseteq Rad\, \kappa_{\mathfrak{g}}$, then $\mathfrak{g}$ is solvable.*

Unlike the case of subalgebras, the restriction of the killing form to an ideal is the killing form on that ideal.

[1] finite dimensional, over an algebraically closed field of characterstic 0

Lemma 5.2.2 *If* $\mathfrak{l} \subseteq \mathfrak{g}$ *is an ideal, then* $\kappa_{\mathfrak{l}} = \kappa_{\mathfrak{g}}|_{\mathfrak{l}}$.

Pf. For $x, y \in \mathfrak{l}$, we have $ad\,x\,ad\,y : \mathfrak{g} \to \mathfrak{l}$, so that $Tr_{\mathfrak{g}}(ad\,x\,ad\,y) = Tr_{\mathfrak{l}}(ad\,x\,ad\,y)$. □

A bilinear form $B : \mathfrak{g} \times \mathfrak{g} \to \mathbb{F}$ is said to be *associative* if $B([x, y], z) = B(x, [y, z])$.

Proposition 5.2.3 *We have*

$$\kappa_{\mathfrak{g}}([x, y],\, z) \;=\; \kappa_{\mathfrak{g}}(x,\, [y, z]). \tag{5.28}$$

Corollary 5.2.4 *The null space of* $\kappa_{\mathfrak{g}}$*,* $Rad\,\kappa_{\mathfrak{g}}$*, is an ideal.*

Pf. If $x \in Rad\,\kappa_{\mathfrak{g}}$ and $y, z \in \mathfrak{g}$, then

$$\kappa([x, y],\, z) \;=\; \kappa(x,\, [y, z]) \;=\; 0 \tag{5.29}$$

so that $[x, y] \in \kappa_{\mathfrak{g}}$ also. □

Lemma 5.2.5 *We have* $Rad\,\kappa_{\mathfrak{g}} \subseteq Rad(\mathfrak{g})$.

Pf. Assume $x \in [Rad\,\kappa_{\mathfrak{g}}, Rad\,\kappa_{\mathfrak{g}}]$ and $y \in Rad\,\kappa_{\mathfrak{g}}$. Because $Rad\,\kappa_{\mathfrak{g}}$ is an ideal,

$$\kappa_{Rad\,\kappa_{\mathfrak{g}}}(x,\, y) \;=\; \kappa_{\mathfrak{g}}(x,\, y) \;=\; 0. \tag{5.30}$$

Thus Cartan's criterion implies $Rad\,\kappa_{\mathfrak{g}}$ is solvable, so by definition $Rad\,\kappa_{\mathfrak{g}} \subseteq Rad(\mathfrak{g})$. □

Theorem 5.2.6 (Humphreys §5.1) *A Lie algebra* $\mathfrak{g}$ *is semisimple if and only if* $\kappa_{\mathfrak{g}}$ *is non-degenerate.*

Pf. If $\mathfrak{g}$ is semisimple, we have $Rad\,\kappa_{\mathfrak{g}} \subseteq Rad(\mathfrak{g}) = \{0\}$, so $\kappa_{\mathfrak{g}}$ is non-degenerate.

If $Rad(\mathfrak{g}) \neq \{0\}$, then $Rad(\mathfrak{g})$ has an abelian ideal. (This can be seen using Lie's theorem, which decomposes $Rad(\mathfrak{g})$ as a flag of ideals of $Rad(\mathfrak{g})$, each of codimension 1 in the next. The "lowest" ideal will then have dimension 1 and therefore be abelian.) Let $\mathfrak{l} \subseteq Rad(\mathfrak{g})$ be this ideal.

Given any $x \in \mathfrak{l}$, $y \in \mathfrak{g}$, we examine the map $ad\,y\,ad\,x : \mathfrak{g} \to \mathfrak{g}$. First, it is clear that $ad\,y\,ad\,x : \mathfrak{g} \to Rad(\mathfrak{g})$, and therefore

$$Tr\,(ad\,y\,ad\,x : \mathfrak{g} \to Rad(\mathfrak{g})) \;=\; Tr\,(ad\,y\,ad\,x : Rad(\mathfrak{g}) \to Rad(\mathfrak{g})) \tag{5.31}$$

However, we actually have $ad\,y\,ad\,x : Rad(\mathfrak{g}) \to \mathfrak{l}$, so

$$Tr\,(ad\,y\,ad\,x : Rad(\mathfrak{g}) \to \mathfrak{l}) \;=\; Tr\,(ad\,y\,ad\,x : \mathfrak{l} \to \mathfrak{l}) \tag{5.32}$$

But $\mathfrak{l}$ is abelian so $ad\,y\,ad\,x : \mathfrak{l} \to \mathfrak{l}$ is the zero operator. □

Chapter 6

Structure of Semisimple Lie Algebras

September 25, 2012

6.1 The abstract Jordan decomposition for semisimple Lie algebras

Recall that a *derivation* δ of a Lie algebra $\mathfrak{g}$ is a map $\delta \in End(V)$ so that $\delta[x,y] = [\delta x, y] + [x, \delta y]$ for all $x, y \in \mathfrak{g}$.

Proposition 6.1.1 *The vector space Der* $\mathfrak{g} \subseteq End(\mathfrak{g})$ *of derivations is a Lie algebra.*

Pf. If $\delta, \delta' \in Der(\mathfrak{g})$ and $x, y \in \mathfrak{g}$ then

$$\begin{aligned}
[\delta,\delta'][x,y] &= \delta\delta'[x,y] - \delta'\delta[x,y] \\
&= \delta([\delta'x,y] + [x,\delta'y]) - \delta'([\delta x,y] + [x,\delta y]) \\
&= [\delta\delta'x,y] + [\delta'x,\delta y] + [\delta x,\delta'y] + [x,\delta\delta'y] - [\delta'\delta x,y] - [\delta x,\delta'y] - [\delta'x,\delta y] - [x,\delta'\delta y] \quad (6.1) \\
&= [\delta\delta'x,y] + [x,\delta\delta'y] - [\delta'\delta x,y] - [x,\delta'\delta y] \\
&= [[\delta,\delta']x,y] + [x,[\delta,\delta]'y].
\end{aligned}$$

Thus $[\delta, \delta']$ is again a derivation. □

Proposition 6.1.2 *If* $\delta \in Der(\mathfrak{g})$*, then* $\delta_s, \delta_n \in Der(\mathfrak{g})$*.*

Pf. Let $\delta \in Der\,\mathfrak{g}$. Because $\mathfrak{F}$ is algebraically closed, there is a Fitting decomposition

$$\mathfrak{g} \;=\; \bigoplus_{i=0}^{n} \mathfrak{g}_{a_i} \tag{6.2}$$

where $\mathfrak{g}_{a_i}$ consists of those $x \in \mathfrak{g}$ with $(\delta - a_i I)^{e_i}.x = 0$ for some $e_i \in \mathbb{N}$. We show that $[\mathfrak{g}_{a_i}, \mathfrak{g}_{a_j}] \subseteq \mathfrak{g}_{a_i+a_j}$. This follows easily using an induction argument and the the computation

$$(\delta \;-\; (a_i + a_j)I)\,.[x, y] \;=\; [(\delta - a_i I)x, y] \;+\; [x, (\delta - a_j I)y]. \tag{6.3}$$

Now assume δ has Jordan decomposition $\delta = \sigma + \nu$ where $\sigma, \nu \in End(\mathfrak{g})$. We know that $y_i \in \mathfrak{g}_{a_i}$ we have $\sigma y_i = a_i y_i$, so that

$$\sigma[x_i,\, x_j] \;=\; (a_i + a_j)[x_i, x_j] \tag{6.4}$$
$$\;=\; [\sigma x_i, x_j] + [x_i, \sigma x_j]. \tag{6.5}$$

Thus σ is a derivation on basis elements, and so on $\mathfrak{g}$. Thus $\sigma, \nu \in Der(\mathfrak{g})$. □

Proposition 6.1.3 *If $\mathfrak{g}$ is a semisimple Lie algebra, then $\mathfrak{g}$ is isomorphic to $Der(\mathfrak{g})$ via the adjoint map $ad : \mathfrak{g} \to Der(\mathfrak{g})$.*

Pf. Because $Z(\mathfrak{g}) \subseteq Rad\,\mathfrak{g} = \{0\}$, the map $ad : \mathfrak{g} \to Der\,\mathfrak{g}$ is injective. An easy computation shows that if $\delta \in Der\,\mathfrak{g}$ and $x \in \mathfrak{g}$ then $[\delta, ad\,x] = ad\,(\delta x)$. Thus $\mathfrak{g}$ is included in $Der(L)$ as an ideal, so that $\kappa_{\mathfrak{g}} = \kappa_{Der\,\mathfrak{g}}\big|_{\mathfrak{g}}$, so that the orthogonal complement $I = \mathfrak{g}^{\perp}$ satisfies $I \cap \mathfrak{g} = \{0\}$. Because I and $\mathfrak{g}$ are non-intersecting ideals, we have $[I, \mathfrak{g}] = \{0\}$, meaning that if $\delta \in I$ we have $0 = [\delta, ad\,x] = ad\,(\delta x)$ for all $x \in \delta$. Since $Ker(ad) = \{0\}$ we have $\delta x = 0$ for all $x \in \mathfrak{g}$, so $\delta = 0$. □

Corollary 6.1.4 *(The abstract Jordan decomposition in semisimple Lie algebras) If $\mathfrak{g}$ is semi-simple and $x \in L$, then there are elements $x_s, x_n \in \mathfrak{g}$ so that $(ad\,x)_s = ad\,x_s$ and $(ad\,x)_n = ad\,x_n$.*

Pf. Combine the previous two propositions. □

Question: Assume $\varphi : \mathfrak{g} \to \mathfrak{gl}(V)$ and $\mathfrak{g}$ is a semisimple Lie algebra. If $x \in \mathfrak{g}$ then it has abstract Jordan decomposition $x = x_s + x_n$, and the image $\varphi(x)$ of x has ordinary Jordan decomposition $\varphi(x) = \varphi(x)_s + \varphi(x)_n$. What is the relationship between $\varphi(x_s)$, $\varphi(x_n)$ and $\varphi(x)_s$, $\varphi(x)_n$?

6.2 Basic structure of semi-simple algebras

We finally come to the relationship between simple Lie algebras (defined as those Lie algebras with non non-trivial ideals) and semisimple Lie algebras (those with no solvable ideals).

Theorem 6.2.1 *If $\mathfrak{g}$ is a semisimple Lie algebra, there are ideals $\mathfrak{g}_1, \ldots, \mathfrak{g}_k \subseteq \mathfrak{g}$ (unique, up to ordering) so that*

$$\mathfrak{g} \;=\; \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_k \tag{6.6}$$

and so that each $\mathfrak{g}_i$ is a simple Lie algebra.

Pf. If $\mathfrak{l}$ is an ideal of $\mathfrak{g}$, then so is $\mathfrak{l}^\perp$. Because $\kappa_{\mathfrak{g}}|_{\mathfrak{l}\cap\mathfrak{l}^\perp} \equiv 0$, we have $\kappa_{\mathfrak{l}\cap\mathfrak{l}^\perp} \equiv 0$, and because $\mathfrak{l}\cap\mathfrak{l}^\perp = Rad\,\kappa_{\mathfrak{l}\cap\mathfrak{l}^\perp} \subseteq Rad(\mathfrak{l}\cap\mathfrak{l}^\perp)$, we have that $\mathfrak{l}\cap\mathfrak{l}^\perp$ is solvable. But any solvable ideal in $\mathfrak{g}$ is trivial, so $\mathfrak{l}\cap\mathfrak{l}^\perp = \{0\}$.

Now we proceed by induction. If $\mathfrak{l}$ is any ideal, it has a complimentary ideal $\mathfrak{l}^\perp$ so that

$$\mathfrak{g} \;=\; \mathfrak{l} \oplus \mathfrak{l}^\perp \tag{6.7}$$

The ideals $\mathfrak{l}$ and $\mathfrak{l}^\perp$ can be similarly subdivided. Note the if $\mathfrak{l}' \subset \mathfrak{l}$ is any ideal of $\mathfrak{l}$, then because $[\mathfrak{l}, \mathfrak{l}^\perp] = 0$, we have that $\mathfrak{l}'$ is an ideal in $\mathfrak{g}$ also. Thus we can continue to subdivide ideals, until we reach simple ideals. Uniqueness is clear. $\square$

Corollary 6.2.2 *If $\mathfrak{g}$ is semisimple, then $\mathfrak{g} = [\mathfrak{g}, \mathfrak{g}]$, and all ideals and homomorphic images of $\mathfrak{g}$ are semisimple. Further, every ideal in $\mathfrak{g}$ is a direct sum of simple ideals. Finally, the adjoint map $ad : \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$ is one-to-one.*

6.3 Modules

Given a Lie algebra $\mathfrak{g}$ and a vector space V, a $\mathfrak{g}$-module on V is a map $\mathfrak{g} \times V \to V$, given on elements by $x.v \in V$ for $x \in \mathfrak{g}$, $v \in V$, that satisfies linearity in both variables, and the usual relation $[x, y].v = x.y.v - y.x.v$. This is the same as a representation.

A $\mathfrak{g}$-module is called irreducible if it has precisely two submodules, itself and $\{0\}$. A module on V is a direct sum of modules $V' \oplus V''$ if V is a direct sum of V' and V'' as a vector space and $\mathfrak{g}.V' \subseteq V'$ and $\mathfrak{g}.V'' \subseteq V''$. A module on V is called *completely reducible* if

$$V \;=\; V_1 \oplus \cdots \oplus V_k \tag{6.8}$$

for some submodules $\{V_i\}_{i=1}^k$.

Chapter 7

Complete Reducibility of Representations of Semisimple Algebras

September 27, 2012

7.1 New modules from old

A few preliminaries are necessary before jumping into the representation theory of semisimple algebras. First a word on creating new $\mathfrak{g}$-modules from old. Any Lie algebra $\mathfrak{g}$ has an action on a 1-dimensional vector space (or $\mathbb{F}$ itself), given by the trivial action. Second, any action on spaces V and W can be extended to an action on $V \otimes W$ by forcing the Leibnitz rule: for any basis vector $v \otimes w \in V \otimes W$ we define

$$x.(v \otimes w) \;=\; x.v \,\otimes\, w \,+\, v \,\otimes\, x.w \tag{7.1}$$

One easily checks that $x.y.(v \otimes w) - y.x.(v \otimes w) = [x, y].(v \otimes w)$. Assuming $\mathfrak{g}$ has an action on V, it has an action on its dual V^* (recall V^* is the vector space of linear functionals $V \to \mathbb{F}$), given by

$$(v.f)(x) \;=\; -f(x.v) \tag{7.2}$$

for any functional $f : V \to \mathbb{F}$ in V^*. This is in fact a version of the "forcing the Leibnitz rule." That is, recalling that we defined $x.(f(v)) = 0$, we define $x.f \in V^*$ implicitly by

$$x.\left(f(v)\right) \;=\; (x.f)(v) \,+\, f(x.v). \tag{7.3}$$

For any vector spaces V, W, we have an isomorphism

$$Hom(V,\, W) \;\approx\; V^* \otimes W, \tag{7.4}$$

so $Hom(V, W)$ is a $\mathfrak{g}$-module whenever V and W are. This can be defined using the above rules for duals and tensor products, or, equivalently, by again forcing the Leibnitz rule: for $F \in Hom(V, W)$, we define $x.F \in Hom(V, W)$ implicitly by

$$x.(F(v)) \;=\; (x.F)(v) \,+\, F(x.v). \tag{7.5}$$

7.2 Schur's lemma and Casimir elements

Theorem 7.2.1 (Schur's Lemma) *If $\mathfrak{g}$ has an irreducible representation on $\mathfrak{gl}(V)$ and if $f \in End(V)$ commutes with every $x \in \mathfrak{g}$, then f is multiplication by a constant.*

Pf. The operator f has a complete eigenspace decomposition, which is preserved by every $x \in \mathfrak{g}$. Namely if $v \in V$ belongs to the generalized eigenspace with eigenvector λ, meaning $(f - \lambda I)^k.v = 0$ for some k, then

$$(f - \lambda I)^k.x.v \;=\; x.(f - \lambda I)^k.v \;=\; 0. \tag{7.6}$$

Thus the generalized λ-eigenspace is preserved by $\mathfrak{g}$ and is therefore a sub-representation. By irreducibility, this must be all of V. Clearly then $f - \lambda I$ is a nilpotent operator on V that commutes with $\mathfrak{g}$. Thus $V_0 = \{v \in V \,|\, (f - \lambda I).v = 0\}$ is non-trivial. But V_0 is preserved by $\mathfrak{g}$, so must equal V. Therefore $f = \lambda I$. □

Now assume V is a $\mathfrak{g}$-module, or specifically that a homomorphism $\varphi : \mathfrak{g} \to \mathfrak{gl}(V)$ exists. As with the adjoint representation we can establish a bilinear form $B_\varphi : \mathfrak{g} \times \mathfrak{g} \to \mathbb{F}$

$$B_\varphi(x,\, y) \;=\; Tr\,(\varphi(x)\,\varphi(y)). \tag{7.7}$$

If φ is the adjoint map, of course this is the Killing form. Clearly

$$B_\varphi([x, y], z) \;=\; B_\varphi(x,\, [y,\, z]) \tag{7.8}$$

so that the radical of B_φ is an ideal of $\mathfrak{g}$. Also, the Cartan criterion implies that the image under φ of the radical of B_φ is solvable.

Thus if φ is a faithful representation of a semisimple algebra, B_φ is non-degenerate. Letting $\{x_i\}_{i=1}^n$ be a basis for $\mathfrak{g}$, a (unique) dual basis $\{y_i\}_{i=1}^n$ exists, meaning the y_i satisfy

$$B_\varphi(x_i,\, y_j) \;=\; \delta_{ij}. \tag{7.9}$$

We define the *casimir element* c_φ of such a representation by

$$c_\varphi \;=\; \sum_{i=1}^n \varphi(x_i)\varphi(y_i) \;\in\; End\, V. \tag{7.10}$$

Lemma 7.2.2 *Given a faithful representation φ of a semisimple Lie algebra, the casimir element commutes with all endomorphisms in $\varphi(\mathfrak{g})$.*

Pf. Let $x \in \mathfrak{g}$ be arbitrary, and define constants

$$\begin{aligned} [x, x_i] &= a_{ij}\, x_j \\ [x, y_i] &= b_{ij}\, x_j \end{aligned} \tag{7.11}$$

We have

$$-b_{ji} \;=\; -\sum_{k=1}^{n} b_{jk}\delta_{ik} \;=\; -B_\varphi(x_i, [x, y_j]) \;=\; B_\varphi([x, x_i],\, y_k) \;=\; \sum_{k=1}^{n} a_{ij}\delta_{jk} \;=\; a_{ij} \tag{7.12}$$

Therefore

$$\begin{aligned} [\varphi(x),\, c_\varphi] &= \sum_{i=1}^{n} [\varphi(x),\, \varphi(x_i)\varphi(y_i)] \\ &= \sum_{i=1}^{n} [\varphi(x),\, \varphi(x_i)]\, \varphi(y_i) \;+\; \sum_{i=1}^{n} \varphi(x_i)\, [\varphi(x),\, \varphi(y_i)] \\ &= \sum_{i=1}^{n} \varphi\left([x,\, x_i]\right) \varphi(y_i) \;+\; \sum_{i=1}^{n} \varphi(x_i)\varphi\left([x,\, y_i]\right) \\ &= \sum_{i,j=1}^{n} a_{ij}\varphi\left(x_j\right) \varphi(y_i) \;+\; \sum_{i,j=1}^{n} b_{ij}\varphi(x_i)\varphi\left(y_j\right) \\ &= 0 \end{aligned} \tag{7.13}$$

□

Lemma 7.2.3 *If $\varphi : \mathfrak{g} \to \mathfrak{gl}(V)$ is an irreducible, faithful representation of the semisimple Lie algebra $\mathfrak{g}$, then the Casimir endomorphism c_φ acts by constant multiplication, with the constant equal to $dim(\mathfrak{g})/dim(V)$.*

Pf. That c_φ acts by constant multiplication by some $\lambda \in \mathbb{F}$ follows from Schur's lemma. We see that

$$Tr(c_\varphi) \;=\; \sum_{i=1}^{dim(\mathfrak{g})} Tr\left(\varphi(x_i)\varphi(y_i)\right) \;=\; \sum_{i=1}^{dim(\mathfrak{g})} B_\varphi(x_i,\, y_i) \;=\; dim(\mathfrak{g}) \tag{7.14}$$

and also that $Tr(c_\varphi) = \lambda \cdot dim(V)$. Thus $\lambda = dim(\mathfrak{g})/dim(V)$. □

7.3 Weyl's Theorem

Lemma 7.3.1 *If $\varphi : \mathfrak{g} \to \mathfrak{gl}(V)$ is a representation and $\mathfrak{g}$ is semisimple, then $\varphi(\mathfrak{g}) \subseteq \mathfrak{sl}(V)$.*

Pf. Because $[\mathfrak{g}, \mathfrak{g}] = \mathfrak{g}$, we have $[\varphi(\mathfrak{g}), \varphi(\mathfrak{g})] = \varphi([\mathfrak{g}, \mathfrak{g}]) = \varphi(\mathfrak{g})$. □

Theorem 7.3.2 (Weyl) *Let $\varphi : \mathfrak{g} \to \mathfrak{gl}(V)$ be a representation*[1] *of a semisimple Lie algebra. Then φ is completely reducible.*

Pf. First, we can assume φ is faithful, for $Ker(\varphi)$ consists of summands on $\mathfrak{g}$, and we can quotient $\mathfrak{g}$ by these summands without affecting the reducibility of the representation.

Step I: *Case of an irreducible codimension 1 submodule*. Assume φ is a representation of $\mathfrak{g}$ on V, and assume $W \subset V$ is an irreducible codimension 1 submodule. The representation on V, being faithful, has a Casimir operator c_φ, which acts by constant multiplication on W (because W is irreducible). In fact $Tr(c_\varphi) = dim(\mathfrak{g}) > 0$. Since V/W is a 1-dimensional module and since $\varphi\mathfrak{g} = [\varphi\mathfrak{g}, \varphi\mathfrak{g}]$ (by the lemma), we have that V/W is a trivial $\mathfrak{g}$-module, so c_φ also acts on V/W by multiplication by 0. All this means that $c_\varphi : V \to V$ has a 1-dimensional Kernel that trivially intersects W, so

$$V \ = \ W \oplus Ker(c_\varphi). \tag{7.15}$$

Since c_φ commutes with $\varphi(\mathfrak{g})$, we have that $Ker(c_\varphi)$ is indeed a (trivial) $\mathfrak{g}$-module.

Step II: *Case of a general codimension 1 irreducible submodule*. Let $W \subset V$ be an arbitrary codimension 1 submodule of $\mathfrak{g}$. If W is not irreducible, there is another submodule $W_1 \subset W$, which we can assume to be maximal. Then W/W_1 is an irreducible submodule of V/W_1, and still has codimension 1. Thus by step I, we have

$$V/W_1 \quad = \quad W/W_1 \oplus V_1/W_1, \tag{7.16}$$

where V_1/W_1 is a 1-dimensional submodule of V/W_1. Because $dim(W) \neq 0$, we have $dim(V_1) < dim(V)$. We also have that W_1 is a codimension 1 submodule of V_1.

Since $dim(V_1) < dim(V)$, an induction argument lets us assert V_1 that $V_1 = W_1 \oplus \mathbb{F}z$, for some $z \in V_1$, as $\mathfrak{g}$-modules. Note that $\mathbb{F}z \cap W = \{0\}$, so $V = W \oplus \mathbb{F}z$ as vector spaces; the question is whether this is a $\mathfrak{g}$-module decomposition. However because $V/W_1 = (W/W_1) \oplus (V_1/W_1)$, we have $\mathfrak{g}.W \subseteq W$, so indeed $W \oplus \mathbb{F}z$ is a $\mathfrak{g}$-module decomposition.

Step III: *The general case*. Assume $W \subset V$ is submodule of strictly smaller dimension, and let $\mathcal{V} \subset Hom(V, W)$ be the subspace of $Hom(V, W)$ consisting of maps that act by constant multiplication on W. Let $\mathcal{W} \subset \mathcal{V}$ be the subset of maps that act as multiplication by zero on W. Moreover, $\mathcal{W} \subset \mathcal{V}$ has codimension, as any element of $\mathcal{V}/\mathcal{W}$ is determined by its scalar action on W.

However we can prove that $\mathcal{V}$ and $\mathcal{W}$ are $\mathfrak{g}$-modules. Letting $F \in \mathcal{V}$, $w \in W$, and $x \in \mathfrak{g}$, we have that $F(w) = \lambda w$ for some $\lambda \in \mathbb{F}$ and, since $x.w \in W$ also $F(x.w) = \lambda x.w$. Thus

$$(x.F)(w) = x.(F(w)) \ - \ F(x.w) \ = \ x.(\lambda w) - \lambda(x.w) \ = \ 0. \tag{7.17}$$

[1]under the usual conditions: $\mathfrak{g}$ and V are finite dimensional, and the field is algebraically closed and of characterstic 0.

Thus all operators in $\mathfrak{g}$ take $\mathcal{V}$ to $\mathcal{W}$, so in particular they are both $\mathfrak{g}$-modules.

By Step II above, there is a $\mathfrak{g}$-submodule in $\mathcal{V}$ complimentary to $\mathcal{W}$, spanned by some operator F_1. Scaling F_1 we can assume $F_1|_W$ is multiplication by 1. Because F_1 generates a 1-dimensional submodules and $\mathfrak{g}$ acts as an element of $\mathfrak{sl}(1,\mathbb{C}) \approx \{0\}$, we have $\mathfrak{g}.F_1 = 0$. Thus we have that $x \in \mathfrak{g}$, $v \in V$ implies

$$0 \;=\; (x.F_1)(v) \;=\; x.(F_1(v)) \;-\; F_1(x.v). \tag{7.18}$$

This is the same as saying F_1 is a $\mathfrak{g}$-module homomorphism $V \to W$. Its kernel is therefore a $\mathfrak{g}$ module, and, since F_1 is the identity on V and maps V to W, must be complimentary as a vector space to W. Therefore

$$V \;=\; W \,\oplus\, Ker(F_1) \tag{7.19}$$

as $\mathfrak{g}$-modules. □

Chapter 8

Preservation of the Jordan Decomposition and Levi's Theorem

Oct 2, 2012

8.1 Preservation of the Jordan decomposition

Theorem 8.1.1 *Assume $\mathfrak{g} \subset \mathfrak{gl}(V)$ is a semisimple linear Lie algebra. Given any $x \in \mathfrak{g}$, the abstract and usual Jordan decompositions coincide.*

Pf. Assume $x_s = P_s(x)$ is the usual Jordan decomposition. Because $Tr(x_s) = Tr(x) = 0$, we know $x_s \in \mathfrak{sl}(V)$. Consider the subalgebra $\mathfrak{g}' \subseteq \mathfrak{gl}(V)$ generated by $\mathfrak{g}$ and x_s. Since $\mathfrak{g}'$ is a $\mathfrak{g}$-module, it is completely reducible by Weyl's theorem, so there is some submodule M with

$$\mathfrak{g}' = \mathfrak{g} \oplus M. \tag{8.1}$$

To see that $M = \{0\}$, let $y \in M$, and note that $[y, \mathfrak{g}] = 0$. Thus y acts by constant multiplication on every irreducible $\mathfrak{g}$-submodule of $W \subseteq V$. Finally we prove that $Tr|_W(y) = 0$, forcing y to be zero.. To see this, first note that $Tr|_W(x_s) = Tr|_W(x) = 0$. The final equality is due to the fact that $x \in [\mathfrak{g}, \mathfrak{g}]$ so is a linear combination of brackets of operators that preserve W, so has trace zero. Then since $\mathfrak{g}'$ is generated, via brackets and linear combinations, of elements of $\mathfrak{g}$ along with x_s, all of which preserve W, we have $Tr|_W(y) = 0$ for all $y \in \mathfrak{g}'$.

This proves that every $y \in M$ acts trivially on every submodule of $\mathfrak{g}$. Thus $y = 0$. □

Corollary 8.1.2 *If φ is a representation of the semisimple Lie algebra $\mathfrak{g}$ on V, then if $x \in \mathfrak{g}$ and $x = x_s + x_n$ is its abstract Jordan decomposition, $\varphi(x) = \varphi(x_s) + \varphi(x_n)$ is the Jordan decomposition of the operator $\varphi(x)$.*

Pf. We have

$$\mathfrak{g} \;=\; \bigoplus_{i=1}^{m} \mathfrak{g}_i \tag{8.2}$$

where the $\mathfrak{g}_i$ are simple. We can write $x = \sum x_i$ uniquely where $x_i \in \mathfrak{g}_i$. Each x_i has its own Jordan decomposition $x_i = x_{i,s} + x_{i,n}$ where $x_{i,s}, x_{i,n} \in \mathfrak{g}_i$. By the mutual commutativity of the $\mathfrak{g}_i$, we therefore have that

$$x \;=\; \sum_{i=1}^{m} x_{i,s} \;+\; \sum_{i=1}^{m} x_{i,n} \tag{8.3}$$

is the Jordan decomposition of x. If φ is a representation, it is faithful on sum subalgebra $\mathfrak{g}' = \sum_{i=1}^{m'} \mathfrak{g}_i$, and has kernel $\mathfrak{g}'' = Ker(\varphi) = \sum_{i=m'+1}^{m} \mathfrak{g}_i$. Writing $x = x' + x''$ where $x' = x_1 + \cdots + x_{m'}$ and $x'' = x_{m'+1} + \cdots + x_m$ we have Jordan decomposition

$$x' \;=\; \sum_{i=1}^{m'} x_{i,s} \;+\; \sum_{i=1}^{m'} x_{i,n}. \tag{8.4}$$

Now φ restricted to $\mathfrak{g}'$ is a faithful representation, so by the previous theorem we have Jordan decomposition

$$\varphi(x) \;=\; \varphi(x') \;=\; \varphi(x'_s) \;+\; \varphi(x'_n). \tag{8.5}$$

But $\varphi(x'_s) = \varphi(x_s)$ and $\varphi(x'_n) = \varphi(x_n)$. □

8.2 Levi's Theorem

Theorem 8.2.1 *Assume $\mathfrak{g}$ is a finite dimensional Lie algebra over an algebraically closed field of characteristic 0, and let $\mathfrak{r}$ be its radical. Then there is a semisimple subalgebra $\mathfrak{l} \subseteq \mathfrak{g}$ so that $\mathfrak{l} \cap \mathfrak{r} = \{0\}$ and $\mathfrak{g} = \mathfrak{r} + \mathfrak{l}$.*

Pf.[1] The proof goes by imitation with the proof of Weyl's theorem.

[1]This proof was taken from Fulton-Harris.

Step I. We first reduce to the case that $\mathfrak{r}$ does not properly contain any ideal of $\mathfrak{g}$. So assuming the theorem is true when $\mathfrak{r}$ contains no ideal that is also an ideal in $\mathfrak{g}$, then let $\mathfrak{a} \subset \mathfrak{l}$ be such an ideal that is maximal in $\mathfrak{r}$. We have

$$\mathfrak{g}/\mathfrak{a} \;=\; \mathfrak{r}/\mathfrak{a} \;+\; \mathfrak{l}/\mathfrak{a} \tag{8.6}$$

where $\mathfrak{l} \subset \mathfrak{g}$ has the properties that $[\mathfrak{l}, \mathfrak{r}] \subseteq \mathfrak{a}$, $\mathfrak{l} \cap \mathfrak{r} \subseteq \mathfrak{a}$, and $\mathfrak{l}/\mathfrak{a}$ is semisimple so $Rad(\mathfrak{l}) = \mathfrak{a}$. One can then make an induction argument on the dimension of $\mathfrak{g}$; since $dim(\mathfrak{l}) < dim(\mathfrak{g})$ we have a splitting $\mathfrak{l} = \mathfrak{a} + \mathfrak{l}'$. Therefore

$$\mathfrak{g} \;=\; \mathfrak{r} + \mathfrak{a} + \mathfrak{l}' \;=\; \mathfrak{r} + \mathfrak{l}' \tag{8.7}$$

since $\mathfrak{a} \subset \mathfrak{r}$.

Step II. We next reduce to the case that, in addition to the conclusion from step I, $\mathfrak{r}$ is also abelian. If $[\mathfrak{r}, \mathfrak{r}] \neq \{0\}$, then actually $\mathfrak{a} = [\mathfrak{r}, \mathfrak{r}] = \mathfrak{r} \cap [\mathfrak{g}, \mathfrak{g}]$ is an ideal in $\mathfrak{g}$ properly contained in $\mathfrak{r}$, so we are back in the situation of Step I.

Step III. In addition to the conclusions from steps I and II, we can reduce to the case that $\mathfrak{r} \neq Z(\mathfrak{g})$. For if that were the case, the adjoint action passes to a natural action of the semisimple algebra $\mathfrak{g}/Z(\mathfrak{g})$ on $\mathfrak{g}$. Since $Z(\mathfrak{g}) \subseteq \mathfrak{g}$ is a submodule, Weyl's theorem gives the existence of a complimentary submodule $\mathfrak{l}$ and

$$\mathfrak{g} \;=\; \mathfrak{l} \oplus Z(\mathfrak{g}) \tag{8.8}$$

where both $\mathfrak{l}$ and of course $Z(\mathfrak{g})$ are ideals.

Step IV. Due to steps I-III, we assume $\mathfrak{r}$ is abelian, has no ideals that are also ideals of $\mathfrak{g}$, and that $\mathfrak{r} \neq Z(\mathfrak{g})$.

Let $V = End(\mathfrak{g})$. We have an action of $\mathfrak{g}$ on V, that is, a map $\mathfrak{g} \to End(End(v))$, defined for any $x \in \mathfrak{g}$, $\varphi \in End(\mathfrak{g})$ by

$$x.\varphi \;=\; [ad\, x,\, \varphi] \;=\; ad\, x \circ \varphi \;-\; \varphi \circ ad\, x. \tag{8.9}$$

This is a Lie algebra homomorphism, so in fact $\mathfrak{g} \to \mathfrak{gl}\,(End(\mathfrak{g}))$.

Now consider the following subspaces of V:

$$\begin{aligned} A \;&=\; \left\{\, ad\, x \;\middle|\; x \in \mathfrak{r} \,\right\} \\ B \;&=\; \left\{\, \varphi \in V \;\middle|\; \varphi : \mathfrak{g} \to \mathfrak{r} \text{ and } \varphi\big|_{\mathfrak{r}} = 0 \,\right\} \\ C \;&=\; \left\{\, \varphi \in V \;\middle|\; \varphi : \mathfrak{g} \to \mathfrak{r} \text{ and } \varphi\big|_{\mathfrak{r}} = const. \,\right\} \end{aligned} \tag{8.10}$$

Note that

$$A \;\subseteq\; B \;\subset\; C \tag{8.11}$$

and $B \subset C$ is a subspace of codimension 1. We claim

$$\begin{aligned} \mathfrak{g}.C \;&\subseteq\; B \\ \mathfrak{r}.C \;&\subseteq\; A. \end{aligned} \tag{8.12}$$

To see these, assume $\varphi \in C$, so that $\varphi(y) = cy$ for some constant c and all $y \in \mathfrak{r}$. Then if $x \in \mathfrak{g}$ and $y \in \mathfrak{r}$ we have

$$\begin{aligned} (x.\varphi)(y) &= (ad\, x)\, \varphi(y) \;-\; \varphi((ad\, x)(y)) \\ &= c\, (ad\, x)\, (y) \;-\; \varphi([x,\, y]) \\ &= c\, [x,\, y] \;-\; c\, [x,\, y] \;=\; 0. \end{aligned} \tag{8.13}$$

so that indeed $\mathfrak{g}.C \subseteq B$. Next assume $x \in \mathfrak{r}$ and $y \in \mathfrak{g}$. Then

$$\begin{aligned} (x.\varphi)(y) &= (ad\, x)\, \varphi(y) \;-\; \varphi((ad\, x)(y)) \\ &= [x,\, \varphi(y)] \;-\; \varphi([x,\, y]) \\ &= 0 \;-\; c\, [x,\, y] \end{aligned} \tag{8.14}$$

where $[x, \varphi(y)] \in [\mathfrak{r}, \mathfrak{r}] = \{0\}$ and $[x, y] \in \mathfrak{r}$ so $\varphi([x, y]) = c[x, y]$. Thus also $\mathfrak{r}.c \subseteq A$.

Now consider the $\mathfrak{g}$-modules $B/A \subseteq C/A$. These both have the property of being cancelled by the radical $\mathfrak{r}$, so these are in fact $\mathfrak{g}/\mathfrak{r}$-modules. Since $\mathfrak{g}/\mathfrak{r}$ is semisimple, Weyl's theorem provides a complimentary submodule D/A (necessarily 1-dimensional) so

$$C/A \;=\; D/A \;+\; B/A. \tag{8.15}$$

Clearly $\mathfrak{g}/\mathfrak{r}$ acts trivially on D/A, so there is a non-zero map $\varphi \in C$ with $\varphi|_{\mathfrak{r}} = 1$ but $\mathfrak{g}.\varphi \in A$. Given this φ, set

$$\mathfrak{l} \;=\; \left\{\, x \in \mathfrak{g} \;\middle|\; x.\varphi = 0 \,\right\}. \tag{8.16}$$

We clearly have $\mathfrak{l}$ as subalgebra of $\mathfrak{g}$. If there is some $x \in \mathfrak{l} \cap \mathfrak{r}$, we have (by the computation above) that

$$0 \;=\; x.\varphi \;=\; -\, ad\, x \tag{8.17}$$

Then x is central, so $\mathbb{F}x$ is an ideal of $\mathfrak{g}$, so we must have $\mathfrak{r} = \mathbb{F}x$. But then $\mathfrak{r} = Z(\mathfrak{g})$, which was assumed not to be the case. Thus $\mathfrak{l} \cap \mathfrak{r} = \{0\}$.

Finally we show that $\mathfrak{g} = \mathfrak{l} \oplus \mathfrak{r}$ in the vector space sense. Because $\mathfrak{g}.\varphi \in A$, if $y \in \mathfrak{g}$ then there is some $y_r \in \mathfrak{r}$ with $y.\varphi = ad\, y_r$. Because $y_r.\varphi = -ad\, y_r$ we have have $(y + y_r).\varphi = 0$ so $y \;=\; (y + y_r) \;-\; y_r$ where $y + y_r \in \mathfrak{l}$ and $-y_r \in \mathfrak{r}$. □

A *reductive* Lie algebra is any Lie algebra that can be written in the form

$$\mathfrak{g} \;=\; \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_n \tag{8.18}$$

where each $\mathfrak{g}_i$ is irreducible under $ad\, \mathfrak{g}$ (meaning each is an ideal in $\mathfrak{g}$ with no proper subideals). It is easy to see that each $\mathfrak{g}_i$ must be simple. Note that, unless $[Rad(\mathfrak{g}), Rad(\mathfrak{g})] = \{0\}$ then the ideal $Rad(\mathfrak{g})$ is never an irreducible $\mathfrak{g}$-module, and also that unless the Levi decomposition

$$\mathfrak{g} \;=\; \mathfrak{l} \;+\; Rad(\mathfrak{g}) \tag{8.19}$$

is a Lie algebra direct sum (meaning $\mathfrak{l}$ is also an ideal), then $\mathfrak{g}$ is not reductive. Thus $\mathfrak{g}$ is reductive if and only if $Rad(\mathfrak{g}) = Z(\mathfrak{g})$.

Chapter 9

Structure of $\mathfrak{sl}(2, \mathbb{C})$-modules

Oct 4, 2012

9.1 Structure of $\mathfrak{sl}(2, \mathbb{C})$-modules

We use the standard basis $\mathfrak{sl}_2 = span\{x, y, h\}$ with

$$[x,\, y] \;=\; h \quad [h,\, x] \;=\; 2x \quad [h,\, y] \;=\; -2y. \tag{9.1}$$

Note that x and y are abstractly nilpotent and h is abstractly semisimple.

Assume the (finite dimensional) vector space V has the structure of a simple module over $\mathfrak{g}$. Because h is abstractly semisimple, it is semisimple as an operator on V and so V decomposes into eigenspaces on which h acts by constant multiplication. Define the λ-weight space

$$V_\lambda \;=\; \left\{\, v \in V \;\middle|\; h.v \;=\; \lambda\, v \,\right\}. \tag{9.2}$$

We have that V is the direct sum of the V_λ that are non-trivial.

If $V_\lambda \neq \{0\}$ we call λ a *weight* of h, and V_λ a *weight space*.

Lemma 9.1.1 *If $v \in V_\lambda$ then $x.v \in V_{\lambda+2}$ and $y.v \in V_{\lambda-2}$.*

Pf. We have

$$\begin{aligned} h.x.v \;&=\; x.h.v \,+\, [h,\, x].v \\ &=\; \lambda\, x.v \,+\, 2x.v \\ &=\; (\lambda+2)\; x.v \end{aligned} \tag{9.3}$$

and similarly for $y.v$. $\square$

From this we easily see that any irreducible $V = \bigoplus V_\lambda$ is generated by any vector $v \in V_\lambda$, and that each V_λ is 1-dimensional.

Lemma 9.1.2 *Let $v_0 \in V_\lambda \subseteq V$ be a maximal element, meaning $x.v_0 = 0$. Set $v_{-1} = 0$ and $v_i = (i!)^{-1} y^i.v_0$. Then*

a) $h.v_i \ = \ (\lambda \ - \ 2i)\, v_i$

b) $y.v_i \ = \ (i+1)\, v_{i+1}$

c) $x.v_i \ = \ (\lambda \ - \ i \ + \ 1)\, v_{i-1}$

Pf. Part (a) is obvious from the lemma, and part (b) is by the definition of the v_i. To prove part (c), assume by induction that $x.v_{i-1} = (\lambda - i + 2)v_{i-2}$

$$\begin{aligned}
x.v_i &= \frac{1}{i}\, x.y.v_{i-1} \\
&= \frac{1}{i}\, ([x,\, y].v_{i-1} \ + \ y.x.v_{i-1}) \\
&= \frac{1}{i}\, (h.v_{i-1} \ + \ y.x.v_{i-1}) \\
&= \frac{1}{i}\, ((\lambda - 2i + 2)\, v_{i-1} \ + \ (\lambda - i + 2)\, y.v_{i-2}) \\
&= \frac{1}{i}\, ((\lambda - 2i + 2)\, v_{i-1} \ + \ (\lambda - i + 2)(i-1)\, v_{i-1}) \\
&= \frac{1}{i}\, (-i \ + \ \lambda - i + 2 \ + \ (\lambda - i + 2)(i-1))\, v_{i-1} \\
&= (\lambda - i + 1)v_{i-1}.
\end{aligned} \tag{9.4}$$

$\square$

Lemma 9.1.3 *The highest weigh λ of h is an integer.*

Pf. Because the representation is finite-dimensional, there is an integer m so that $v_m \neq 0$ but $v_{m+1} = \frac{1}{m+1} y.v_m = 0$. Then

$$0 \ = \ x.v_{m+1} \ = \ (\lambda \ - \ (m+1) + 1) \ v_m \tag{9.5}$$

so that $\lambda = m$ because $v_m \neq 0$. $\square$

To summarize:

Theorem 9.1.4 *Let V be an irreducible module for $\mathfrak{sl}(2, \mathbb{C})$.*

a) If $\dim V = m+1$ *then* V *is the direct sum of weight spaces* V_μ *for* $\mu = m, m-2, \ldots, -m$

b) V *has, up to scalar multiplication, a unique maximal vector, whose weight is* m

c) The action of $\mathfrak{sl}(2,\mathbb{C})$ *on* V *is as described above. In particular, there is precisely one irreducible* $\mathfrak{sl}(2,\mathbb{C})$*-module of dimension* $m+1$, $m \geq 0$.

One notices that the abstractly nilpotent operators x, y are nilpotent as operators on V_λ as well. Similarly h is abstractly semisimple, and acts as a semisimple operator on V_λ.

9.2 A word on the Casimir operator

We already know the that on the irreducible module V_λ, the Casimir operator is $\frac{3}{\lambda+1}$, but let us see this in action. Letting V_λ be the irreducible module of dimension $\lambda+1$, and letting

$$v_i = (i!)^{-1} y^i . v_0 \tag{9.6}$$

we have

$$x. = \begin{pmatrix} 0 & \lambda & 0 & 0 & & 0 & 0 \\ 0 & 0 & \lambda-1 & 0 & \ldots & 0 & 0 \\ 0 & 0 & 0 & \lambda-2 & & 0 & 0 \\ 0 & 0 & 0 & 0 & & 0 & 0 \\ & \vdots & & & & \vdots & \\ 0 & 0 & 0 & 0 & \ldots & 0 & 1 \\ 0 & 0 & 0 & 0 & & 0 & 0 \end{pmatrix} \tag{9.7}$$

$$y. = \begin{pmatrix} 0 & 0 & 0 & 0 & & 0 & 0 \\ 1 & 0 & 0 & 0 & \ldots & 0 & 0 \\ 0 & 2 & 0 & 0 & & 0 & 0 \\ 0 & 0 & 3 & 0 & & 0 & 0 \\ & \vdots & & & & \vdots & \\ 0 & 0 & 0 & 0 & \ldots & 0 & 0 \\ 0 & 0 & 0 & 0 & & \lambda & 0 \end{pmatrix} \tag{9.8}$$

$$h. = \begin{pmatrix} \lambda & 0 & 0 & 0 & & 0 & 0 \\ 0 & \lambda-2 & 0 & 0 & \ldots & 0 & 0 \\ 0 & 0 & \lambda-3 & 0 & & 0 & 0 \\ 0 & 0 & 0 & \lambda-4 & & 0 & 0 \\ & \vdots & & & & \vdots & \\ 0 & 0 & 0 & 0 & \ldots & -\lambda+2 & 0 \\ 0 & 0 & 0 & 0 & & 0 & -\lambda \end{pmatrix} \tag{9.9}$$

From these we easily compute

$$x.y. \;=\; \begin{pmatrix} \lambda & 0 & & & & & \\ 0 & 2(\lambda-1) & & & & & \\ & & \ddots & & & & \\ & & & i(\lambda-i+1) & & & \\ & & & & \ddots & & \\ & & & & & \lambda & 0 \\ & & & & & 0 & 0 \end{pmatrix} \tag{9.10}$$

$$y.x. \;=\; \begin{pmatrix} 0 & 0 & & & & & \\ 0 & \lambda & & & & & \\ & & \ddots & & & & \\ & & & (i-1)(\lambda-i+2) & & & \\ & & & & \ddots & & \\ & & & & & 2(\lambda-1) & 0 \\ & & & & & 0 & \lambda \end{pmatrix} \tag{9.11}$$

$$h.h. \;=\; \begin{pmatrix} \lambda^2 & 0 & & & & & \\ 0 & (\lambda-2)^2 & & & & & \\ & & \ddots & & & & \\ & & & (\lambda-2i)^2 & & & \\ & & & & \ddots & & \\ & & & & & (\lambda-2)^2 & 0 \\ & & & & & 0 & \lambda^2 \end{pmatrix} \tag{9.12}$$

The representation's bilinear form is then

$$B_V \;=\; \begin{pmatrix} 0 & \frac{1}{6}\lambda(\lambda+2)(\lambda+1) & 0 \\ \frac{1}{6}\lambda(\lambda+2)(\lambda+1) & 0 & 0 \\ 0 & 0 & \frac{1}{3}\lambda(\lambda+2)(\lambda+1) \end{pmatrix} \tag{9.13}$$

and the Casimir operator is therefore

$$C_V \;=\; \frac{1}{\frac{1}{6}\lambda(\lambda+2)(\lambda+1)}\left(x.y. + y.x. + \frac{1}{2}h.h.\right) \tag{9.14}$$

We easily compute that $x.y + y.x + \frac{1}{2}h.h$ is the $(\lambda+1)\,by\,(\lambda+1)$ diagonal matrix with $\frac{1}{2}\lambda(\lambda+2)$ on the diagonal. We therefore have

$$C_V \;=\; \frac{\frac{1}{2}\lambda(\lambda+2)}{\frac{1}{6}\lambda(\lambda+2)(\lambda+1)} \;=\; \frac{3}{\lambda+1}. \tag{9.15}$$

Now the intrinsic (aka Killing) bilinear form is

$$\kappa_{\mathfrak{sl}(2,\mathbb{C})} \;=\; \begin{pmatrix} 0 & 4 & 0 \\ 4 & 0 & 0 \\ 0 & 0 & 8 \end{pmatrix} \tag{9.16}$$

so the intrinsic Casimir operator is

$$\frac{1}{4}\left(x.y. \,+\, y.x. \,+\, \frac{1}{2}h.h.\right). \tag{9.17}$$

which operates on V_λ by

$$\frac{1}{8}\lambda\,(\lambda+2)\,. \tag{9.18}$$

For various reasons we will multiply by 2, and obtain the operator

$$C_{\mathfrak{g}} \;=\; \frac{1}{2}\left(x.y + y.x. + \frac{1}{2}h.h.\right) \tag{9.19}$$

which acts on V_λ by

$$\frac{1}{4}\lambda\,(\lambda+2)\,. \tag{9.20}$$

9.3 A word on notation

Although the real lie algebras $\mathfrak{sl}_2$ and $\mathfrak{o}(3)$ are distinct, we have see that an isomorphism

$$\mathfrak{sl}_2 \otimes \mathbb{C} \;\approx\; \mathfrak{o}(3) \otimes \mathbb{C} \tag{9.21}$$

exists. In those applications for which this is a dominant consideration, it is standard to use an $\mathfrak{sl}_2$-basis for which

$$[x,\,y] \;=\; 2h \qquad [h,x] \;=\; x \qquad [h,y] \;=\; -y. \tag{9.22}$$

Basically the old h has been replaced by the vector of half its length. The notation for modules is also a little different: the irreducible module with highest weight l is denoted

$$V_l \tag{9.23}$$

where now l takes on integer and half-integer values. Also, the weight-spaces that comprise V_l are denoted V_l^m where V_l^m is a 1-dimensional weight space of weight m, and

$$V_l \;=\; \bigoplus_{m=-l}^{l} V_l^m \tag{9.24}$$

(the summation increments by 1 even when l is half-integral). The creation and annihilation operators now raise and lower weights by 1 instead of 2, meaning $x.V_l^m = V_l^{m+1}$ (unless $m = l$) and $y.V_l^m = V_l^{m-1}$ (unless $m = -l$).

The Killing form is slightly different

$$\kappa = \begin{pmatrix} 0 & 4 & 0 \\ 4 & 0 & 0 \\ 0 & 0 & 2 \end{pmatrix}. \tag{9.25}$$

The abstract Casimir operator is then

$$\frac{1}{4}\,(x.y. + y.x. + 2h.h.) = 2l(l+1) \tag{9.26}$$

(note that with $\lambda = 2l$ we have $\frac{1}{2}\lambda(\lambda+2) = 2l(l+1)$). Again it is common to multiply by $\frac{1}{2}$ and use the following expression for the abstract Casimir operator:

$$C_V = \frac{1}{2}\,(x.y. + y.x. + 2h.h.) = l(l+1). \tag{9.27}$$

To see the significance of this, consider the $\mathfrak{o}(3)$-basis X, Y, Z for which $[X, Y] = Z$ and cyclic permutations. The isomorphism is

$$\begin{aligned} x &= -Y + \sqrt{-1}\,X \\ y &= Y + \sqrt{-1}\,X \\ h &= \sqrt{-1}\,Z \end{aligned} \tag{9.28}$$

and the Casimir (9.27) has the very natural expression

$$C_V = -X.X. - Y.Y. - Z.Z. \tag{9.29}$$

which, incidentally, is half the intrinsic Casimir on $\mathfrak{o}(3)$.

Chapter 10

Examples of $\mathfrak{sl}(2, \mathbb{C})$-modules

Oct 9, 2012

This chapter was not taken from Humphrey's book. Rather, it is a description of a concrete—and extremely famous—use of the thoery so far: computing quantum numbers for the Hydrogen atom.

Let $\mathcal{H} = L^2(\mathbb{R}^3)$ be the space of complex-valued L^2-functions on $\mathbb{R}^3$, and let $\mathcal{S} = L^2(\mathbb{S}^2)$ be the space of complex-valued L^2-functions on $\mathbb{S}^2$.

10.1 Preliminaries: Riemannian geometry of $\mathbb{S}^2$

To coordinatize the sphere of radius r, $\mathbb{S}^2(r) \subset \mathbb{R}^3$, we use stereographic projection. The coordinates (u, v) on $\mathbb{R}^2$ are given in terms of the $\mathbb{R}^3$-coordinates (x, y, z) by

$$u = \frac{r\,x}{r - z} \qquad v = \frac{r\,x}{r - z}. \tag{10.1}$$

Conversely, the old coordinates (x, y, z) are now functions on $\mathbb{S}^2$, given by

$$\begin{aligned} x &= \frac{2\,r^2\,u}{u^2 + v^2 + r^2} \\ y &= \frac{2\,r^2\,v}{u^2 + v^2 + r^2} \\ z &= \frac{r(u^2 + v^2 - r^2)}{u^2 + v^2 + r^2}. \end{aligned} \tag{10.2}$$

We compute the metric

$$g \;=\; \left(\frac{2r^2}{u^2+v^2+r^2}\right)^2 (du\otimes du \,+\, dv\otimes dv)\,. \tag{10.3}$$

In particular we have the Laplacian operator

$$\triangle_{\mathbb{S}^2(r)} \;=\; \left(\frac{u^2+v^2+r^2}{2r^2}\right)^2\left(\frac{\partial^2}{\partial u^2}\,+\,\frac{\partial^2}{\partial v^2}\right) \,-\, \frac{u^2+v^2+r^2}{r^4}\left(u\,\frac{\partial}{\partial u}\,+\,v\,\frac{\partial}{\partial v}\right). \tag{10.4}$$

10.2 Action of $SO(3)$ and $\mathfrak{so}(3)$ on $\mathcal{H}$

There is a natural representation of $SO(3)$ on this space: if $G\in SO(3)$, we define

$$(G.f)(x,\,y,\,z) \;=\; f(G.(x,\,y,\,z)) \tag{10.5}$$

where the action on (x,y,z) is by left multiplication, or

$$G.(x,\,y,\,z) \;=\; \begin{pmatrix} G_{11} & G_{12} & G_{13}\\ G_{21} & G_{22} & G_{23}\\ G_{31} & G_{32} & G_{33}\end{pmatrix}\begin{pmatrix} x\\ y\\ z\end{pmatrix}. \tag{10.6}$$

Let

$$\begin{aligned} G_x(\theta) \;&=\; \begin{pmatrix} 1 & 0 & 0\\ 0 & \cos\theta & -\sin\theta\\ 0 & \sin\theta & \cos\theta\end{pmatrix}\\ G_y(\theta) \;&=\; \begin{pmatrix} \cos\theta & 0 & \sin\theta\\ 0 & 1 & 0\\ -\sin\theta & 0 & \cos\theta\end{pmatrix}\\ G_z(\theta) \;&=\; \begin{pmatrix} \cos\theta & -\sin\theta & 0\\ \sin\theta & \cos\theta & 0\\ 0 & 0 & 1\end{pmatrix}. \end{aligned} \tag{10.7}$$

be the principle rotations. Then we have that $\mathfrak{so}(3) = span_{\mathbb{R}}\{X,Y,Z\}$ where as usual

$$X \;=\; \frac{\partial}{\partial\theta}\Big|_{\theta=0} G_x(\theta) \qquad Y \;=\; \frac{\partial}{\partial\theta}\Big|_{\theta=0} G_y(\theta) \qquad Z \;=\; \frac{\partial}{\partial\theta}\Big|_{\theta=0} G_z(\theta). \tag{10.8}$$

To compute the action of X, Y, Z on $\mathcal{H}$, we simply compute, for any $f=f(x,y,z)\in\mathcal{H}$, that

$$(X.f)(x,y,z) \;=\; \frac{\partial}{\partial\theta}\Big|_{\theta=0}(G_x.f)(x,y,z) \;=\; \left(y\frac{\partial f}{\partial z}\,-\,z\frac{\partial f}{\partial y}\right) \tag{10.9}$$

and so on, so that

$$X = \left(y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y}\right)$$
$$Y = \left(z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z}\right) \tag{10.10}$$
$$Z = \left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right).$$

Note that the rotations themselves can now be expressed as differential operators. For example rotation about the z-axis by an angle of θ is

$$G_z(\theta).f = e^{\theta(x\partial_y - y\partial_x)}f. \tag{10.11}$$

In the past we have discussed the existence of a Lie algebra isomorphism $\mathfrak{so}(3,\mathbb{C}) \approx \mathfrak{sl}(2,\mathbb{C})$. This can be realized by

$$\begin{aligned} J_+ &= -Y + \sqrt{-1}X \\ &= -\left(z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z}\right) + \sqrt{-1}\left(y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y}\right) \\ J_- &= Y + \sqrt{-1}X \\ &= \left(z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z}\right) + \sqrt{-1}\left(y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y}\right) \\ J_z &= \sqrt{-1}Z \\ &= \sqrt{-1}\left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right). \end{aligned} \tag{10.12}$$

In this basis, the brackets are

$$[J_+,\, J_-] = 2J_z \qquad [J_z,\, J_+] = J_+ \qquad [J_z,\, J_-] = -J_- \tag{10.13}$$

and the Killing form is

$$\kappa_{\mathfrak{sl}(2)} = \begin{pmatrix} 0 & 4 & 0 \\ 4 & 0 & 0 \\ 0 & 0 & 2 \end{pmatrix}. \tag{10.14}$$

Thus the abstract Casimir operator is

$$C_{\mathfrak{g}} = \frac{1}{2}\left(J_+J_- + J_-J_+ + 2J_zJ_z\right) = -XX - YY - ZZ. \tag{10.15}$$

We compute

$$\begin{aligned} C_g = &-(y^2+z^2)\frac{\partial^2}{\partial x^2} + -(x^2+z^2)\frac{\partial^2}{\partial y^2} + -(x^2+y^2)\frac{\partial^2}{\partial z^2} \\ &+ 2yz\frac{\partial^2}{\partial y\partial z} + 2xz\frac{\partial^2}{\partial x\partial z} + 2xy\frac{\partial^2}{\partial x\partial y} \\ &+ 2x\frac{\partial}{\partial x} + 2y\frac{\partial}{\partial y} + 2z\frac{\partial}{\partial z}. \end{aligned} \tag{10.16}$$

Unfortunately this is an unusable mess, so let's try our luck elsewhere.

10.2.1 Action of $SO(3)$ on $\mathcal{S}$

The action of $SO(3)$ on $\mathbb{R}^3$ restricts to $\mathbb{S}^2(r)$, so there is a natural action of $SO(3)$ on $\mathcal{S} = L^2(\mathbb{S}^2)$. Although writing it down is bulky, the action of $\mathfrak{so}(3)$ is

$$\begin{aligned} X &= \frac{uv}{r}\frac{\partial}{\partial u} - \frac{u^2 - v^2 - r^2}{2r}\frac{\partial}{\partial v} \\ Y &= \frac{-u^2 + v^2 - r^2}{2r}\frac{\partial}{\partial u} - \frac{uv}{r}\frac{\partial}{\partial v} \\ Z &= u\frac{\partial}{\partial v} - v\frac{\partial}{\partial u}. \end{aligned} \tag{10.17}$$

Once again we form the $\mathfrak{sl}_2$-basis

$$J_+ = -Y + \sqrt{-1}X \qquad J_- = -Y - \sqrt{-1}X \qquad J_z = \sqrt{-1}Z. \tag{10.18}$$

In computing the Casimir, we have to be a bit careful, due to the non-trivial geometry. Computing just J_+J_+ for instance will not give a covariant object. The actual Casimir is

$$C_{\mathfrak{g}} = -XX - YY - ZZ + \nabla_X X + \nabla_Y Y + \nabla_Z Z. \tag{10.19}$$

Worked out, we obtain

$$\begin{aligned} C_{\mathfrak{g}} &= -r^2\left(\frac{u^2+v^2+r^2}{2r^2}\right)^2\left(\frac{\partial^2}{\partial u^2} + \frac{\partial^2}{\partial v^2}\right) + r^2\left(\frac{u^2+v^2+r^2}{r^4}\right)\left(u\frac{\partial}{\partial u} + v\frac{\partial}{\partial v}\right) \\ &= -r^2\triangle_{\mathbb{S}^2(r)} \end{aligned} \tag{10.20}$$

or r^2 times the Laplacian.

10.3 Example: Spherical Harmonics

The operators J_+, J_-, J_z commute with the Casimir operator $C = \triangle_{\mathbb{S}^2}$, so we gain a great deal of information on the spectrum of $\triangle_{\mathbb{S}^2}$.

From Fredholm theory we know that the multiplicity of each eigenvalue of $\triangle_{\mathbb{S}^2}$ is finite, and that the span of the eigenvalues is a basis for $\mathcal{S}$. Representation theory gives us further structure on the eigenspaces. We have seen that the eigenvalues of $\triangle_{\mathbb{S}^2(r)}$ must be

$$-\frac{1}{r^2}\,l(l+1) \tag{10.21}$$

for integral or half-integral values of $l \geq 0$. Further, any function $f \in \mathcal{H}$ with $\triangle_{\mathbb{S}^2} f = -r^{-2}l(l+1)f$ generates an eigenspace of dimension $2l+1$, which is produced by powers of

J_+ and J_- acting on f. Finally, if $f_1, f_2 \in \mathcal{H}$ both have $\triangle_{\mathbb{S}^2} f_i = -l(l+1)f_i$ and $J_z.f_i = 0$, it is easy to prove $f_1 = f_2$. This is because f_1, f_2 are rotationally symmetric, so the Poisson equation on $\mathbb{S}^2$ reduces to an ODE (2nd order linear) with two fixed boundary conditions, so has a unique solution. Thus the eigenspace corresponding to the eigenvalue $l(l+1)$ is irreducible and has dimension precisely $2l+1$.

Eigenvalues corresponding to half-integral l can be ruled out, as the resulting $\mathfrak{o}(3)$ representation does not pass to an $SO(3)$ (although it does pass to an $SU(2)$). On the other hand, every integral l corresponds to an eigenvalue. This is because a function f with $Z.f = 0$ and $\triangle f = -l(l+1)$ can be constructed (by reducing to a second order linear ODE and using some Sturm-Liouville theory).

We have shows that for given integers l, m where $l \geq 0$ and $m \in \{-l, -l+1, \dots, l-1, l\}$, there is a unique function

$$Y_l^m : \mathbb{S}^2 \;\to\; \mathbb{C} \tag{10.22}$$

that has norm 1, is an $l(l+1)$-eigenvalue of $-\triangle_{\mathbb{S}^2}$, and is an m-eigenvalue of J_z. These are known as the (complex) *spherical harmonics*. Note, for instance, that $Y_{0,0} = (Vol\, \mathbb{S}^2)^{-1}$ is a constant.

10.4 Quantum Numbers

Consider the Hamiltonian for a particle trapped in a $\frac{1}{r^2}$-field:

$$H \;=\; -\frac{\hbar}{2m}\triangle \;-\; \frac{k}{r} \tag{10.23}$$

for some constant α. Rechosing units, of course we have

$$H \;=\; -\triangle \;-\; r^{-1}. \tag{10.24}$$

In Quantum mechanics we look for solutions to the Schrödinger equation

$$\sqrt{-1}\hbar\frac{\partial}{\partial t}\varphi \;=\; H\varphi \tag{10.25}$$

of the form $\psi : \mathbb{R}^3 \to \mathbb{C}$ in $\mathcal{H} = L^2(\mathbb{R}^3)$. To do so, we find $\psi : \mathbb{R}^3 \to \mathbb{C}$ with the property $H\psi_E = E\psi_E$ for some constant E (the energy), which means

$$e^{-\frac{\sqrt{-1}}{\hbar}Et}\psi_E \;:\; \mathbb{R}^{1+3} \to \mathbb{C} \tag{10.26}$$

satisfies (10.25), and create a superposition

$$\psi \;=\; \int \lambda(E)\, e^{-\frac{\sqrt{-1}}{\hbar}Et}\psi_E\, dE \tag{10.27}$$

where $\lambda(E)$ is some function.

To solve $E\psi = H\psi$, we have

$$\triangle\,\psi \,+\, r^{-1}\psi \,+\, E\,\psi \;=\; 0. \tag{10.28}$$

Since

$$\triangle_{\mathbb{R}^3} = \frac{\partial^2}{\partial r^2} \,+\, \frac{2}{r}\frac{\partial}{\partial r} \,+\, \triangle_{\mathbb{S}^3(r)} \tag{10.29}$$

where $\triangle_{\mathbb{S}^3(r)}$ is the intrinsic Laplacian on the 2-sphere of radius r.

Decomposing $\mathcal{H} = \bigoplus \mathcal{H}_l$ where $\mathcal{H}_l$ is spanned by functions of the form $f(r)Y_l^m$ for $-l \leq m \leq l$, we have that $\psi \in \mathcal{H}_l$, $\psi = f(r)Y_l^m$, satisfies

$$\left(f''(r) \,+\, \frac{2}{r}f'(r) \,+\, \left(\frac{1}{r} \,+\, E \,-\, \frac{l(l+1)}{r^2}\right) f(r)\right) Y_l^m \;=\; 0 \tag{10.30}$$

The remainder of the theory follows from quite classical analysis, which we will not go into except to say that the ODE inside the larger parantheses is well-studied, and it is known that bounded solutions only occur when $l+1-\frac{1}{\sqrt{-E}}$ is a non-positive integer, meaning the allowable values for E are

$$-\frac{1}{(l+1)^2} \;\leq\; E = -\frac{1}{n^2} \;<\; 0 \tag{10.31}$$

so in particular, we have the condition

$$(l+1)^2 \;\leq\; n^2 = -\frac{1}{E} \tag{10.32}$$

which corresponds with the intuition that E is the electron's potential energy, has a definite lower bound, and approaches zero energy (as $n \to \infty$) when it reaches its unbound state.

Thus the three integers n, l, m (aka "quantum numbers") completely characterize solutions to the eigenvalue problem $E\varphi = H\varphi$. These are the *principle quantum number* $n = 1/\sqrt{-E}$ with $n \in \{1, 2, \ldots\}$, the *azimuthal* (or total angular momentum) quantum number l (with $l \in \{0, 1, \ldots, n-1\}$), and the *magnetic quantum number* m (with $m \in \{-l, -l+1, \ldots, l-1, l\}$).

Of course in high school we learned that four quantum numbers exist; the missing number is spin. However spin has no classical analog so cannot be ascertained via $SO(3)$ representations on 3-space (and in addition requires the Dirac, not the Schrödinger, equation, along with the representations for half-integral l which we threw away).

Chapter 11

Cartan Subalgebras

October 11, 2012

11.1 Maximal Toral Subalgebras

A *toral subalgebra* of a Lie algebra $\mathfrak{g}$ is any subalgebra consisting entirely of abstractly semisimple elements.

Lemma 11.1.1 *If $\mathfrak{t} \subset \mathfrak{g}$ is any toral subalgebra, then $\mathfrak{t}$ is abelian.*

Pf. Assume $x \in \mathfrak{t}$ has $(ad\, x)|_{\mathfrak{t}} \neq 0$. Because x is semisimple, it is ad-simisimple, so there is some $y \in \mathfrak{t}$ with $(ad\, x)(y) = ay$. Note that

$$0 \;=\; (ad\, y)(ad\, x)(y) \;=\; -(ad\, y)^2 x. \tag{11.1}$$

However, y is also semisimple, so we can write

$$x \;=\; x_1 \,+\, \ldots \,+\, x_n \tag{11.2}$$

where $(ad\, y)(x_i) = \lambda_i x_i$. Therefore

$$0 \;=\; (ad\, y)^2 x \;=\; \sum_{i=1}^n \lambda_i^2 x_i \tag{11.3}$$

so that each $\lambda_i = 0$. However this contradicts

$$0 \;\neq\; ay \;=\; (ad\, x)y \;=\; -(ad\, y)x \;=\; \sum_{i=1}^n \lambda_i x_i. \tag{11.4}$$

□

If $\mathfrak{g}$ is a semisimple Lie algebra, then any maximal toral subalgebra $\mathfrak{h}$ is called a *Cartan subalgebra* or CSA for short. Caution: in the non-semisimple case, this is not the proper use of the term CSA.

Since $\mathfrak{h}$ is a commuting subalgebra, the action of $ad_{\mathfrak{g}}\mathfrak{h}$ is simultaneously diagonalizable. This means that $\mathfrak{g}$ has a complete root-space decomposition: there are finitely many linear functionals $\alpha \in \mathfrak{h}^* \triangleq \Lambda$, $\alpha : \mathfrak{h} \to \mathbb{F}$ so that

$$\mathfrak{g} = \bigoplus_{\alpha \in \Lambda} \mathfrak{g}_\alpha \tag{11.5}$$

where

$$\mathfrak{g}_\alpha \;=\; \{\, x \in \mathfrak{g} \mid h.x \;=\; \alpha(h)\, x \text{ for all } h \in \mathfrak{h} \,\} \tag{11.6}$$

is the (non-trivial) weight-space associated to the functional α. Note that $\mathfrak{g}_0$ is $C_{\mathfrak{g}}(\mathfrak{h})$, the centralizer of $\mathfrak{h}$ in $\mathfrak{g}$.

Proposition 11.1.2 *Assume $\mathfrak{g}$ is a semisimple Lie algebra. Then*

a) Given $\alpha, \beta \in \mathfrak{h}^*$, we have $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \in \mathfrak{g}_{\alpha+\beta}$.

b) If $x \in \mathfrak{g}_\alpha$ then $ad\, x$ is nilpotent.

c) If $\beta \neq -\alpha$ then L_α is perpendicular to L_β under the Killing form.

d) The restriction of $ad_{\mathfrak{g}}\mathfrak{h}$ to $\mathfrak{g}_0$ is non-degenerate.

Pf. Let $x_\alpha \in \mathfrak{g}_\alpha$, $x_\beta \in \mathfrak{g}_\beta$. For (a), we have

$$\begin{aligned} [h, [x_\alpha, x_\beta]] &= [[h, x_\alpha], x_\beta] + [x_\alpha, [h, x_\beta]] && (11.7)\\ &= \alpha(h)[x_\alpha, x_\beta] \;+\; \beta(h)[x_\alpha, x_\beta] \;=\; (\alpha+\beta)(h)\,[x_\alpha, x_\beta]. && (11.8) \end{aligned}$$

For (b), note that $(ad\, x_\alpha)^n : \mathfrak{g}_\beta \to \mathfrak{g}_{\beta+n\alpha}$. Since the number of weight spaces is finite, there is some n so $(ad\, x_\alpha)^n = 0$. For (c), by associativity of κ we have

$$0 \;=\; \kappa([h, x_\alpha], x_\beta) \;+\; \kappa(x_\alpha, [h, x_\beta]) \;=\; (\alpha(h) + \beta(h))\, \kappa(x_\alpha, x_\beta). \tag{11.9}$$

For (d), we know that $ad_{\mathfrak{g}}\mathfrak{h} : \mathfrak{g} \to \mathfrak{g}$ is non-degenerate, but also that $ad_{\mathfrak{g}}\mathfrak{h}\big|_{\mathfrak{g}_\alpha} = 0$ unless $\alpha = 0$. Thus $ad_{\mathfrak{g}}\mathfrak{h}$ must be non-degenerate on $\mathfrak{h}$. □

Proposition 11.1.3 *We have $\mathfrak{g}_0 = \mathfrak{h}$.*

Pf. Write $C = C_{\mathfrak{g}}(\mathfrak{h}) = \mathfrak{g}_0$.

Step I. *C contains the abstract semisimple and nilpotent parts of all its elements.* Since $x \in C$ has $ad\, x : \mathfrak{h} \to 0$ and $(ad\, x)_s = ad\, x_s$, $(ad\, x)_n = ad\, x_n$ are given as polynomials in $ad\, x$ without constant term, so also $(ad\, x)_s, (ad\, x)_n : \mathfrak{h} \to 0$ meaning that $x_s, x_n \in C$.

Step II. *All semisimple elements of C lie in $\mathfrak{h}$.* If x is semisimple and $x : \mathfrak{h} \to 0$ then $\mathfrak{h} + \mathbb{F}x$ is both an algebra and is toral.

Step III. *The restriction of κ to $\mathfrak{h}$ is non-degenerate.* If $x \in C$ is nilpotent, then because x commutes with $\mathfrak{h}$ we have $\kappa(x, \mathfrak{h}) = 0$. But $C = \mathfrak{h} + \textit{nilpotents}$ so if $h \in \mathfrak{h}$, its dual must be in $\mathfrak{h}$.

Step IV. *C is nilpotent.* Since $C = \mathfrak{h} + \textit{nilpotents}$ and C commutes with $\mathfrak{h}$, any element of C is a sum of something in $\mathfrak{h}$ with a nilpotent, and since everything commutes, and arbitrary element is nilpotent. Thus C, being ad-nilpotent, is nilpotent.

Step V. *We have $\mathfrak{h} \cap [C, C] = \{0\}$.* We have $\kappa(\mathfrak{h}, [C, C]) = \kappa([C, \mathfrak{h}], C) = 0$. Thus $[C, C]$ intersects $\mathfrak{h}$ trivially.

Step VI. *C is abelian.* If $D = [C, C]$ is non-trivial, then any $x \in D$ is nilpotent. Also $D \cap Z(C)$ is nontrivial, so we can assume $x \in D \cap Z(C)$. But then $x_n \in Z(C)$ is non-zero, so $\kappa(x_n, C) = 0$, which is impossible because κ restricted to C is non-degenerate.

Step VII. If $x \in C \setminus H$ then its nilpotent part $x_n \in C$ commutes with C, and so $\kappa(x_n, C) = 0$, which is impossible because $\kappa|_C$ is nondegenerate. □

We can now identify $\mathfrak{h}$ with $\mathfrak{h}^*$ via $\kappa_{\mathfrak{g}}$. Given any linear functional $\alpha \in \mathfrak{h}^*$, define $t_\alpha \in \mathfrak{h}$ by

$$\alpha(h) \;=\; \kappa(t_\alpha,\, h) \tag{11.10}$$

for all $h \in \mathfrak{h}$.

Proposition 11.1.4 *Assume $\mathfrak{g}$ is a finite dimensional simple Lie algebra over an algebraically closed field of characteristic 0. Let Φ be the set of roots of the adjoint action of $\mathfrak{h}$ on $\mathfrak{g}$. Then*

a) Φ spans $\mathfrak{h}^*$.

b) If $\alpha \in \Phi$ then $-\alpha \in \Phi$.

c) If $\alpha \in \Phi$ and $x \in \mathfrak{g}_\alpha$, $y \in \mathfrak{g}_{-\alpha}$, then $[x, y] = \kappa(x, y)\, t_\alpha$ (recall t_α is the κ-dual of α).

d) If $\alpha \in \Phi$ then $[\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}] \subseteq \mathfrak{h}$ is 1-dimensional.

e) If $\alpha \in \Phi$ then $\alpha(t_\alpha) = \kappa(t_\alpha, t_\alpha) \neq 0$.

f) If $\alpha \in \Phi$ and $x_\alpha \in \mathfrak{g}_\alpha$, there is some $y_\alpha \in \mathfrak{g}_{-\alpha}$ so that setting $h_\alpha = [x_\alpha, y_\alpha]$ we have $span_{\mathbb{F}}\{x_\alpha, y_\alpha, h_\alpha\} \approx \mathfrak{sl}(2, \mathbb{F})$.

g) Given $x_\alpha \in \mathfrak{g}_\alpha$, the choice of y_α in (f) leads to $h_\alpha = \frac{2t_\alpha}{\kappa(t_\alpha, t_\alpha)}$. In addition $h_\alpha = -h_{-\alpha}$.

Pf. (a). If not, there is some $h \in \mathfrak{h}$ so that $\alpha(h) = 0$ for all $\alpha \in \Phi$. But then $[h, \mathfrak{g}_\alpha] = 0$ for all $\alpha \in \Phi$, so $h \in Z(\mathfrak{g})$, an impossibility.

(b). If $\alpha \in \Phi$ but $-\alpha \notin \Phi$ then $[\mathfrak{g}_\alpha, \mathfrak{g}] = \{0\}$, meaning $\mathfrak{g}_\alpha \in Z(\mathfrak{g})$, again an impossibility.

(c). Given $h \in \mathfrak{h}$, $x \in \mathfrak{g}_\alpha$, $y \in \mathfrak{g}_{-\alpha}$, the associativity of κ implies

$$\begin{aligned} \kappa(h, [x, y]) &= \kappa([h, x], y) \\ &= \alpha(h)\, \kappa(x, y) \\ &= \kappa(t_\alpha, h)\, \kappa(x, y) \\ &= \kappa(t_\alpha \kappa(x, y), h) \end{aligned} \tag{11.11}$$

but then $([x, y] - \kappa(x, y)t_\alpha) \in \mathfrak{h}$ and $h \perp ([x, y] - \kappa(x, y)t_\alpha)$ for all $h \in \mathfrak{h}$, forcing $[x, y] = \kappa(x, y)t_\alpha$.

(d). Follows directly from (c).

(e). Assume $\alpha(t_\alpha) = \kappa(t_\alpha, t_\alpha) = 0$. Then $\mathfrak{s} = span_{\mathbb{C}}\{x, y, t_\alpha)$ is a nilpotent Lie algebra. Consider its *ad*-representation on $\mathfrak{g}$.

(f) and (g). Given any $x_\alpha \in L_\alpha$, pick $y_\alpha \in L_{-\alpha}$ so that $\kappa(x_\alpha, y_\alpha) = \frac{2\, t_\alpha}{\kappa(t_{\alpha}, t_\alpha)} \triangleq h_\alpha$. Then $[h_\alpha, x_\alpha] = \alpha(h_\alpha)x_\alpha = 2x_\alpha$ and $[h_\alpha, y_\alpha] = -\alpha(h_\alpha)y_\alpha = -2y_\alpha$, so we have our copy of $\mathfrak{sl}(2, \mathbb{C})$. □

Chapter 12

Root Space Decomposition

October 16, 2012

12.1 Review

First let us recall the situation. Let $\mathfrak{g}$ be a simple algebra, with maximal toral subalgebra $\mathfrak{h}$ (which we are calling a CSA, or Cartan Subalgebra). We have that $\mathfrak{h}$ acts on $\mathfrak{g}$ via the adjoint action, and since $\mathfrak{h}$ has only mutually commuting, abstractly semisimple elements, we have that the action of $\mathfrak{h}$ is simultaneously diagonalizable. Thus $\mathfrak{g}$ decomposes into weight spaces, called in this special case *root spaces*:

$$\begin{aligned} \mathfrak{g} &= \bigoplus_{\alpha \in \Phi \cup \{0\}} \mathfrak{g}_\alpha \\ \mathfrak{g}_\alpha &= \{ \, x \in \mathfrak{g} \mid h.x = \alpha(h)\, x \text{ for all } h \in \mathfrak{h} \, \} . \end{aligned} \tag{12.1}$$

We defined Φ to be the set of roots of $\mathfrak{g}$ relative to the choice of $\mathfrak{h}$, or in other words, the non-zero weights for the adjoint action of $\mathfrak{h}$ on $\mathfrak{g}$. We proved:

i) $\mathfrak{h} = C_{\mathfrak{g}}(\mathfrak{h}) = \mathfrak{g}_0$

ii) Φ spans $\mathfrak{h}^*$

iii) $\alpha \in \Phi$ implies $-\alpha \in \Phi$

iv) $[\mathfrak{g}_\alpha, \, \mathfrak{g}_\beta] \in \mathfrak{g}_{\alpha+\beta}$

v) $x_\alpha \in \mathfrak{g}_\alpha$, $y_\alpha \in \mathfrak{g}_{-\alpha}$ implies $[x_\alpha, y_\alpha] = \kappa_{\mathfrak{g}}(x_\alpha, y_\alpha) t_\alpha$ (where t_α is the $\kappa_{\mathfrak{g}}$-dual of α)

vi) $[\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}] = \mathbb{F}\, t_\alpha \subseteq \mathfrak{h}$

vii) If $x_\alpha \in \mathfrak{g}_\alpha$ then $y_\alpha \in \mathfrak{g}_{-\alpha}$ exists so $\{x_\alpha, y_\alpha, h_\alpha\}$ is a standard basis for some $\mathfrak{sl}(2,\mathbb{C}) \subseteq \mathfrak{g}$

viii) $h_\alpha = \frac{2t_\alpha}{\kappa(t_\alpha,t_\alpha)}$ and $h_\alpha = -h_{-\alpha}$

Since κ is nondegenerate, we can define an inner product on the dual space $\mathfrak{h}^*$ directly by

$$(\alpha,\, \beta) \;=\; \kappa(t_\alpha,\, t_\beta) \tag{12.2}$$

where, recall, t_α (resp. t_β) is the dual of α under κ.

12.2 Structure of the root set

Given any $x_\alpha \in \mathfrak{g}_\alpha$, it spans a copy of $\mathfrak{sl}(2,\mathbb{C})$. We will use S_α to denote this copy of $\mathfrak{sl}(2,\mathbb{C})$. There is no reason, presently, to think S_α is uniquely defined, as some other choice of $x \in \mathfrak{g}_\alpha$ could result in a different S_α, although we know they would have the same h_α.

Proposition 12.2.1 *The root space $\mathfrak{g}_\alpha$ is 1-dimensional; in particular the choice of S_α is unique. Further, if $\alpha \in \Phi$, then $c\alpha \in \Phi$ if and only if $c \in \{1,-1\}$.*

Pf. Picking any $x_\alpha \in \mathfrak{g}_\alpha$, let S_α be the $\mathfrak{sl}_2$ subalgebra $\{x_\alpha, y_\alpha, h_\alpha\}$ guaranteed by (*vii*).

Let $\mathfrak{r}_\alpha \subseteq \mathfrak{g}$ be

$$\mathfrak{r}_\alpha \;=\; span_{\mathbb{F}}\,\{\,\mathfrak{g}_{c\alpha} \mid c \in \mathbb{F}\,\}\,. \tag{12.3}$$

By definition of $\mathfrak{g}_\alpha$ we have that $h_\alpha : \mathfrak{g}_{c\alpha} \to \mathfrak{g}_{c\alpha}$ acts by

$$h_\alpha(x) \;=\; c\alpha(h_\alpha) \;=\; 2c, \tag{12.4}$$

or multiplication by $2c$. However note that $\mathfrak{r}_\alpha$ is itself an S_α-module, so h_α acts by integral multiplication on all elements. Thus $c \in \{\pm\frac{1}{2}n \,|\, n \in \mathbb{Z}\}$.

Further, by (*i*) above, $h_\alpha.x = 0$ (in other words $[h_\alpha, x] = 0$) only when $x \in \mathfrak{h}$. Now $Ker(\alpha)$ is a codimension 1 subspace of $\mathfrak{h}$ on which S_α acts trivially. Thus if $\mathfrak{t}$ is an irreducible submodule of $\mathfrak{r}_\alpha$ of odd dimension (so it has a vector space of weight zero), it is generated by h_α acted on by S_α. But the module generated by the action of S_α on h_α is just S_α itself.

Now suppose $\mathfrak{t} \subset \mathfrak{r}_\alpha$ is an irreducible submodule of even dimension. There must be a weight space $\mathfrak{g}_{\frac{1}{2}\alpha}$, so let $x_{\frac{1}{2}\alpha} \in \mathfrak{g}_{\frac{1}{2}\alpha}$. But by (*vii*) above, there is some $y_{\frac{1}{2}\alpha} \in \mathfrak{g}_{-\frac{1}{2}\alpha}$ and an $h_{\frac{1}{2}\alpha} \in \mathfrak{h}$ so that $\{x_{\frac{1}{2}\alpha}, y_{\frac{1}{2}\alpha}, h_{\frac{1}{2}\alpha}\}$ spans a copy of $\mathfrak{sl}(2,\mathbb{C})$, which we can denote $S_{\frac{1}{2}\alpha}$. But then the original $\mathfrak{r}_\alpha$ is an $S_{\frac{1}{2}\alpha}$-module. However the subspace $S_\alpha \subseteq \mathfrak{r}$ generates (via the action of $S_{\frac{1}{2}\alpha}$) an irreducible module of odd weight, so by the previous paragraph, S_α must be precisely $S_{\frac{1}{2}\alpha}$, an impossibility. □

In the proof of Proposition 12.2.1 we examined the action of S_α on the span of all root spaces of the form $\mathfrak{g}_{c\alpha}$, and proved a nice structure theorem (that $span_{\mathbb{C}}\{g_{c\alpha} \,|\, c \in \mathbb{C}\} = S_\alpha$). But if $\beta \in \Phi$ is a distinct root, meaning $\beta \neq \pm\alpha$, how does S_α act on the various $\mathfrak{g}_\beta$?

Chapter 13

Root Space Decomposition II

October 18, 2012

13.1 Review

First let us recall the situation. Let $\mathfrak{g}$ be a simple algebra, with maximal toral subalgebra $\mathfrak{h}$ (which we are calling a CSA, or Cartan Subalgebra). We have that $\mathfrak{h}$ acts on $\mathfrak{g}$ via the adjoint action, and since $\mathfrak{h}$ has only mutually commuting, abstractly semisimple elements, we have that the action of $\mathfrak{h}$ is simultaneously diagonalizable. Thus $\mathfrak{g}$ decomposes into weight spaces, called in this special case *root spaces*:

$$\begin{aligned} \mathfrak{g} &= \bigoplus_{\alpha \in \Phi \cup \{0\}} \mathfrak{g}_\alpha \\ \mathfrak{g}_\alpha &= \{\, x \in \mathfrak{g} \mid h.x = \alpha(h)\, x \text{ for all } h \in \mathfrak{h} \,\}. \end{aligned} \tag{13.1}$$

We defined Φ to be the set of roots of $\mathfrak{g}$ relative to the choice of $\mathfrak{h}$, or in other words, the non-zero weights for the adjoint action of $\mathfrak{h}$ on $\mathfrak{g}$. We proved:

i) $\mathfrak{h} = C_{\mathfrak{g}}(\mathfrak{h}) = \mathfrak{g}_0$

ii) Φ spans $\mathfrak{h}^*$

iii) $\alpha \in \Phi$ implies $-\alpha \in \Phi$

iv) $[\mathfrak{g}_\alpha,\, \mathfrak{g}_\beta] \in \mathfrak{g}_{\alpha+\beta}$

v) $x_\alpha \in \mathfrak{g}_\alpha$, $y_\alpha \in \mathfrak{g}_{-\alpha}$ implies $[x_\alpha, y_\alpha] = \kappa_{\mathfrak{g}}(x_\alpha, y_\alpha) t_\alpha$ (where t_α is the $\kappa_{\mathfrak{g}}$-dual of α)

vi) $[\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}] = \mathbb{F}\, t_\alpha \subseteq \mathfrak{h}$

vii) If $x_\alpha \in \mathfrak{g}_\alpha$ then $y_\alpha \in \mathfrak{g}_{-\alpha}$ exists so $\{x_\alpha, y_\alpha, h_\alpha\}$ is a standard basis for some $\mathfrak{sl}(2,\mathbb{C}) \subseteq \mathfrak{g}$

viii) $h_\alpha = \frac{2t_\alpha}{\kappa(t_\alpha,t_\alpha)}$ and $h_\alpha = -h_{-\alpha}$

ix) $\kappa_\mathfrak{g}|_\mathfrak{h}$ is positive definite.

Since κ is nondegenerate, we can define an inner product on the dual space $\mathfrak{h}^*$ directly by

$$(\alpha,\,\beta) \;=\; \kappa(t_\alpha,\,t_\beta) \tag{13.2}$$

where, recall, t_α (resp. t_β) is the dual of α under κ.

Lemma 13.1.1 *Assume α and β are roots, and that $\alpha + c\beta$ are all roots where $c \in \mathbb{C}$. Then*

- *The number $\frac{2(\alpha,\beta)}{(\alpha,\alpha)}$ is an integer*
- *c is an integer*
- *The direct sum of root spaces of the form $\mathfrak{g}_{\beta+i\alpha}$ is an irreducible S_α module. In particular if $x_\alpha \in \mathfrak{g}$, we have that $(ad\,x_\alpha)^i : \mathfrak{g}_\alpha \to \mathfrak{g}_{\alpha+i\beta}$ is an isomorphism.*

Pf. The vector space

$$T \;=\; \bigoplus \mathfrak{g}_{\beta+c\alpha} \tag{13.3}$$

where c ranges over all numbers in $\mathbb{C}$ so $\beta + c\alpha \in \Phi$, is a finite dimensional S_α-module, and each $\mathfrak{g}_{\beta+i\alpha}$ is 1-dimensional. The weight (in the $\mathfrak{sl}_2$-sense) of the space $\mathfrak{g}_{\beta+c\alpha}$ is computed by selecting some $x \in \mathfrak{g}_{\alpha+c\beta}$ and computing $(ad\,h_\alpha)x$. We have

$$(ad\,h_\alpha)x \;=\; (\beta+c\alpha)(h_\alpha)\,x \;=\; \left(\frac{2\,(\alpha,\beta)}{(\alpha,\alpha)} \,+\, 2c\right) x \tag{13.4}$$

Since weights must be integers, we can take $c = 0$ to obtain the integrality of $\frac{2(\alpha,\beta)}{(\alpha,\alpha)}$. Then we also have that $2c \in \mathbb{Z}$.

Now we rule out c being half-integral. Assume $c = \frac{n}{2}$ (n odd). We can always choose n so $\mathfrak{g}_{\beta+\frac{n}{2}\alpha}$ has weight 0 or else weight 1 (with respect to the weight operator h_α). We have

$$h_\alpha . x_{\beta+\frac{n}{2}\alpha} \;=\; \left(\frac{2(\alpha,\beta)}{(\alpha,\alpha)} \,+\, n\right) x_{\beta+\frac{n}{2}\alpha} \tag{13.5}$$

so $\frac{2(\alpha,\beta)}{(\alpha,\alpha)}$ is $-n$ or $1-n$.

Case I: Assume the S_α-weight of $\mathfrak{g}_{\beta+\frac{n}{2}\alpha}$ is 0, so $\frac{2(\alpha,\beta)}{(\alpha,\alpha)} = -n$.
Consider the action of $S_{\beta+\frac{1}{2}\alpha} \approx \mathfrak{sl}_2$. We have

$$h_{\beta+\frac{n}{2}\alpha} . x_\beta \;\;=\;\; \beta(h_{\beta+\frac{n}{2}\alpha})\,x_\alpha \;=\; \frac{2(\beta,\beta) \,+\, n(\alpha,\beta)}{(\beta,\beta)+n(\alpha,\beta)+\frac{n^2}{4}(\alpha,\alpha)}\,x_\alpha \tag{13.6}$$

so with $(\alpha,\alpha) = -\frac{2}{n}(\alpha,\beta)$ we have

$$h_{\beta+\frac{1}{2}\alpha}.x_\beta \quad = \quad 2\,x_\alpha \tag{13.7}$$

Therefore $(ad\,y_{\beta+\frac{n}{2}\alpha})(x_\beta)$ is non-zero, and lies in $\mathfrak{g}_{-\frac{n}{2}\alpha}$. This is impossible because $-\frac{n}{2}\alpha$ is not a root.

Case II: Assume the S_α-weight of $\mathfrak{g}_{\beta+\frac{n}{2}\alpha}$ is 1, so $\frac{2(\alpha,\beta)}{(\alpha,\alpha)} = 1-n$.
First note that since n is odd, the right side is even so that $\frac{\alpha\cdot\beta}{\alpha\cdot\alpha}$ is an integer. Applying a destruction operator, we get that $\mathfrak{g}_{\beta+\frac{n-2}{2}\alpha}$ is also a non-trivial weight space. Since β is a root, we have the algebra S_β, and we can find the S_β-weights of $\mathfrak{g}_{\beta+\frac{n}{2}\alpha}$, $\mathfrak{g}_{\beta+\frac{n-2}{2}\alpha}$ as follows:

$$\begin{aligned} h_\beta.x_{\beta+\frac{n}{2}\alpha} &= \left(\beta+\frac{n}{2}\alpha\right)\cdot\frac{2\beta}{\beta\cdot\beta}\,x_{\beta+\frac{n}{2}\alpha} \\ &= 2\left(1+\frac{n}{2}\frac{\alpha\cdot\beta}{\beta\cdot\beta}\right)\,x_{\beta+\frac{n}{2}\alpha}. \end{aligned} \tag{13.8}$$

Since $2+n\frac{\alpha\cdot\beta}{\beta\cdot\beta}$ is an integer, n is odd, and $2\frac{\alpha\cdot\beta}{\beta\cdot\beta}$ is an integer, we have that $\frac{\alpha\cdot\beta}{\beta\cdot\beta}$ is an integer. Now we cheat a bit and use the theorem below, which states that κ is positive definite (note that this is not cyclical: that result does not use this one). By Cauchy-Schwarz we have

$$\frac{(\alpha\cdot\beta)^2}{(\alpha\cdot\alpha)(\beta\cdot\beta)} \leq 1 \tag{13.9}$$

so either $\alpha\cdot\beta = 0$, or α and β are parallel. We assumed they are not parallel, so $\alpha\cdot\beta = 0$, meaning the calculation above gives

$$h_\beta.x_{\beta+\frac{n}{2}\alpha} \quad = \quad 2\,x_{\beta+\frac{n}{2}\alpha}. \tag{13.10}$$

Therefore we can apply a destruction operator (namely y_β) to obtain a non-trivial weight space, namely $g_{\frac{n}{2}\alpha}$. Yet this is impossible because $\frac{n}{2} \neq \pm 1$.

The lemma's final assertion is a direct consequence of the fact that $\beta + i\alpha$ is a root only when $i \in \mathbb{Z}$ and that each root space is 1-dimensional. □

The numbers $\frac{2(\beta,\alpha)}{(\alpha,\alpha)}$ are called the *Cartan integers*. The set of roots of the form $\beta + i\alpha$ is called the α-string through β.

Proposition 13.1.2 *Assume $\alpha, \beta \in \Phi$ and $\beta \neq \pm\alpha$ Then*

a) The S_α-weight of $\mathfrak{g}_\beta$ is $\frac{2(\beta,\alpha)}{(\alpha,\alpha)}$.

b) If β is a maximal weight for S_α, then the S_α module generated by $\mathfrak{g}_\beta$ is

$$\mathfrak{g}_\beta \oplus \mathfrak{g}_{\beta-\alpha} \oplus \cdots \oplus \mathfrak{g}_{\beta-\frac{2(\alpha,\beta)}{(\alpha,\alpha)}\alpha} \tag{13.11}$$

c) If α, β *are any roots, then* $\beta - \frac{2(\beta,\alpha)}{(\alpha,\alpha)}\alpha \in \mathfrak{h}^*$ *is a root.*

Pf. For (a), we compute

$$(ad\, h_\alpha)(x_\beta) \;=\; \beta(h_\alpha)\, x_\beta \;=\; \left(t_\beta,\, \frac{2\, t_\alpha}{(\alpha,\alpha)}\right) x_\beta \;=\; \frac{2(\alpha,\beta)}{(\alpha,\alpha)}\, x_\beta. \tag{13.12}$$

For (b), (c), let β be a root. The weight of the (possibly trivial) root space $g_{\beta - \frac{2(\alpha,\beta)}{(\alpha,\alpha)}\alpha}$ is

$$\left(\beta - \frac{2(\alpha,\beta)}{(\alpha,\alpha)}\alpha\right)(h_\alpha) \;=\; \beta(h_\alpha) - \frac{2(\alpha,\beta)}{(\alpha,\alpha)}\alpha(h_\alpha) \;=\; -\frac{2(\alpha,\beta)}{(\alpha,\alpha)} \tag{13.13}$$

which is the negative of the weight of the root space of $\mathfrak{g}_\beta$. Since the negative of a weight is a weight, we have that $\mathfrak{g}_{\beta - \frac{2(\alpha,\beta)}{(\alpha,\alpha)}\alpha}$ must be non-trivial. □

13.2 The Euclidean space E and its inner product κ

Lemma 13.2.1 *If* $\beta, \gamma \in \mathfrak{h}^*$ *then* $(\beta,\gamma) = \sum_{\alpha\in\Phi}(\alpha,\beta)(\alpha,\gamma)$.

Pf. If $\Phi = \{\alpha_1, \ldots, \alpha_m\}$ then the decomposition

$$\mathfrak{g} \;=\; \mathfrak{h} \oplus \bigoplus_{\alpha_i \in \Phi} \mathfrak{g}_{\alpha_i} \tag{13.14}$$

diagonalizes the adjoint action of all elements of $\mathfrak{h}$. In fact if $h \in \mathfrak{h}$ then

$$ad\, h \;=\; \begin{pmatrix} 0 & & & & & \\ & \ddots & & & & \\ & & 0 & & & \\ & & & \alpha_1(h) & & \\ & & & & \ddots & \\ & & & & & \alpha_m(h) \end{pmatrix} \tag{13.15}$$

Thus

$$\begin{aligned} (\beta,\, \gamma) \;&=\; \kappa(t_\beta,\, t_\gamma) \\ &=\; Tr\, ad\, t_\beta\, ad\, t_\gamma \\ &=\; \sum_{i=1}^m \alpha_i(t_\beta)\alpha_i(t_\gamma) \\ &=\; \sum_{i=1}^m (\beta,\, \alpha_i)\,(\gamma,\, \alpha_i)\,. \end{aligned} \tag{13.16}$$

□

Lemma 13.2.2 *Let $\{\alpha_1, \ldots, \alpha_n\} \subseteq \Phi$ be a $\mathbb{C}$-basis of $\mathfrak{h}^*$. Then $\Phi \subset span_{\mathbb{Q}}\{\alpha_1, \ldots, \alpha_n\}$.*

Pf. We have $\beta = \sum_{i=1}^n c_i\alpha_i$ for some constants $c_i \in \mathbb{C}$. Then $(\beta, \alpha_j) = \sum_i c_i A_{ij}$ where $A_{ij} = (\alpha_i, \alpha_j)$. Note that A_{ij} is invertible. Then

$$\frac{2(\beta, \alpha_j)}{(\alpha_j, \alpha_j)} = \sum_{i=1}^n c_i \frac{2(\alpha_i, \alpha_j)}{(\alpha_j, \alpha_j)} \tag{13.17}$$

Since $M_{ij} = A_{ij}(\alpha_j, \alpha_j)^{-1}$ is invertible and all numbers besides the c_i are integers, the c_i are rationals. □

Lemma 13.2.3 *If $\beta, \gamma \in \Phi$, then (β, γ) is rational, and $(\beta, \beta) > 0$.*

Pf. We have $(\beta, \beta) = \sum_{\alpha\in\Phi}(\alpha, \beta)^2$ so that

$$\frac{4}{(\beta, \beta)} = \sum_{i=1}^m \left(\frac{2(\alpha, \beta)}{(\beta, \beta)}\right)^2. \tag{13.18}$$

The numbers on the right are all integers, so (β, β) is rational. Now letting $\gamma \in \Phi$ we have

$$\begin{aligned} (\beta, \gamma) &= \sum_{\alpha\in\Phi}(\alpha, \beta)(\alpha, \gamma) \\ \frac{4(\beta, \gamma)}{(\beta, \beta)(\gamma, \gamma)} &= \sum_{\alpha\in\Phi} \frac{2(\alpha, \beta)}{(\beta, \beta)} \frac{2(\alpha, \gamma)}{(\gamma, \gamma)} \end{aligned} \tag{13.19}$$

where the right-hand side is integral, and (β, β) and (γ, γ) are rational. Therefore (β, γ) is rational. Finally with $(\beta, \beta) = \sum_{\alpha\in\Phi}(\alpha, \beta)^2$ again, we see that (β, β) is the sum of non-negative rationals. Thus κ is positive semi-definite. Since it is non-degenerate, it is therefore positive definite and $(\beta, \beta) > 0$. □

Theorem 13.2.4 *Setting $E = span_{\mathbb{R}}\Phi$ and restricting κ to E, we have that $dim_{\mathbb{R}}E = dim_{\mathbb{C}}\mathfrak{h}$ and κ is positive definite inner product.*

Pf. Trivlal. □

13.3 Root Space Axioms

It is useful to put some of our conclusions into one place; the theorem that follows verifies what we will call the *root space axioms.*

Theorem 13.3.1 *Let $\mathfrak{g}$ be a semisimple Lie algebra, $\mathfrak{h}$ any maximal toral subalgebra, Φ the set of roots associated to $\mathfrak{h}$, and $E = span_{\mathbb{R}}\Phi$ with positive definite inner product κ. Then*

a) Φ *spans* E*, and* $0 \notin \Phi$

b) If $\alpha \in \Phi$ *then* $-\alpha \in \Phi$*, but* $c\alpha \notin \Phi$ *for* $c \neq \pm 1$

c) If $\alpha, \beta \in \Phi$*, then* $\beta - \frac{2(\beta,\alpha)}{(\alpha,\alpha)}\alpha \in \Phi$

d) If $\alpha, \beta \in \Phi$ *then* $\frac{2(\beta,\alpha)}{(\alpha,\alpha)} \in \mathbb{Z}$

□

Chapter 14

$\mathfrak{o}(4)$ and $\mathfrak{g}_2$

October 23, 2012

In this lecture we take a closer look at the orthogonal algebras.

14.1 Example: $\mathfrak{o}(4)$

14.1.1 Identification with an alternating algebra

Given a Riemannian metric $g(\cdot,\,\cdot)$ on any vector space V, there is are two a bilinear maps

$$\begin{aligned} G : V^{\otimes 2} \otimes V^{\otimes 2} \;&\to\; \mathbb{C} \\ H : V^{\otimes 2} \otimes V^{\otimes 2} \;&\to\; V^{\otimes 2}. \end{aligned} \tag{14.1}$$

The "metric" G is directly inherited from the metric on V. Namely, on basis elements

$$G(e_i \otimes e_j,\, e_k \otimes e_l) \;=\; g(e_i,\, e_k)\, g(e_j,\, e_l) \tag{14.2}$$

On $\bigwedge^2 V$, it is conventional to divide by 2:

$$\begin{aligned} G(e_i \wedge e_j,\; e_k \wedge e_l) \;&=\; \frac{1}{2} G(e_i \otimes e_j - e_j \otimes e_i,\; e_k \otimes e_l - e_l \otimes e_k) \\ &=\; g(e_i,\, e_k) g(e_j, e_l) \;-\; g(e_i,\, e_l) g(e_j, e_k). \end{aligned} \tag{14.3}$$

The second map, H is given by contraction on middle terms:

$$H\left(e_i \otimes e_j,\, e_k \otimes e_l\right) \;=\; e_i \otimes e_l \cdot g\left(e_j,\, e_k\right). \tag{14.4}$$

This passes to $\bigwedge^2 V$, which becomes a Lie algebra under the bracket:

$$[e_i \wedge e_j,\, e_k \wedge e_l] \;=\; H(e_i \wedge e_j,\, e_k \wedge e_l) \,-\, H(e_k \wedge e_l,\, e_i \wedge e_j). \tag{14.5}$$

If $V = span_{\mathbb{C}}\{e_1, \ldots, e_n\}$ is $\mathbb{R}^n$, then $\bigwedge^2 V$, with this bracket, is isomorphic to $\mathfrak{sl}(n, \mathbb{C})$.

14.1.2 The 4-dimensional case

The 4-dimensional case is special, because there exists a second non-degenerate, bilinear, symmetric form. If $V = \{e_1, e_2, e_3, e_4\}$ is $\mathbb{R}^4$, then define

$$B : \bigwedge^2 V \otimes \bigwedge^2 V \to \mathbb{C} \tag{14.6}$$

on homogeneous elements by

$$B(a_i \wedge a_j,\, a_k \wedge a_l) \;=\; \frac{Det(A_{ij})}{\sqrt{|Det(E_{ij})|}} \tag{14.7}$$

where $A_{ij} = g(e_i, a_j)$ and $B_{ij} = g(e_i, e_j)$. One clearly sees that this definition is bilinear, symmetric, and independent of the choice of basis, as long as the change retains the orientation. If e_1, e_2, e_3, e_4 is an orthonormal basis, we can abuse notation and set

$$B(a_i \wedge a_j,\, a_k \wedge a_l) \;=\; \frac{a_1 \wedge a_2 \wedge a_3 \wedge a_4}{e_1 \wedge e_2 \wedge e_3 \wedge e_4}. \tag{14.8}$$

It is easy to verify non-degeneracy; since $\bigwedge^2 V$ is 6-dimensional, one can check this on a basis.

Thus a unitary linear operator $* : \bigwedge^2 V \to \bigwedge^2 V$, known as the duality operator or Hodge star, can be defined implicitly by

$$B\,(\, v \wedge w,\, *(v \wedge w)\,) \;=\; G(\, v \wedge w\,,\; v \wedge w\,). \tag{14.9}$$

By the bilinearity of both factors, we have

$$** \;=\; Id \,:\, \bigwedge^2 V \;\to\; \bigwedge^2 V. \tag{14.10}$$

Note that if e_1, e_2, e_3, e_4 is an ordered, orthonormal basis, then we have as usual

$$\begin{array}{llll} *(e_1 \wedge e_2) \;=\; e_3 \wedge e_4 & *(e_3 \wedge e_4) \;=\; e_1 \wedge e_2 \\ *(e_1 \wedge e_3) \;=\; -\, e_2 \wedge e_4 & *(e_2 \wedge e_4) \;=\; -e_1 \wedge e_3 \\ *(e_1 \wedge e_4) \;=\; e_2 \wedge e_3 & *(e_2 \wedge e_3) \;=\; e_1 \wedge e_4. \end{array} \tag{14.11}$$

We have thus established a map

$$* \,:\, \mathfrak{o}(4) \;\to\; \mathfrak{o}(4) \tag{14.12}$$

with

$$**\ =\ 1. \tag{14.13}$$

The possible eigenvalues of $*$ are therefore ± 1. These can denoted by

$$\begin{aligned} \bigwedge^{+} V \ &=\ \mathfrak{o}^{+}(4)\ =\ +1 \text{ eigenspace of } * \\ \bigwedge^{-} V \ &=\ \mathfrak{o}^{-}(4)\ =\ -1 \text{ eigenspace of } * \end{aligned} \tag{14.14}$$

Further, it can be proved that

$$*[v \wedge w,\, a \wedge b]\ =\ [*(v \wedge w),\, a \wedge b]. \tag{14.15}$$

From this and the semi-simplicity of $\mathfrak{o}(4)$ it follows that

$$\begin{aligned} \left[\,\mathfrak{o}^{+}(4),\, \mathfrak{o}^{+}(4)\,\right] \ &=\ \mathfrak{o}^{+}(4) \\ \left[\,\mathfrak{o}^{-}(4),\, \mathfrak{o}^{-}(4)\,\right] \ &=\ \mathfrak{o}^{-}(4) \\ \left[\,\mathfrak{o}^{+}(4),\, \mathfrak{o}^{-}(4)\,\right] \ &=\ \{0\}. \end{aligned} \tag{14.16}$$

In particular $\mathfrak{o}(4)$ is not simple:

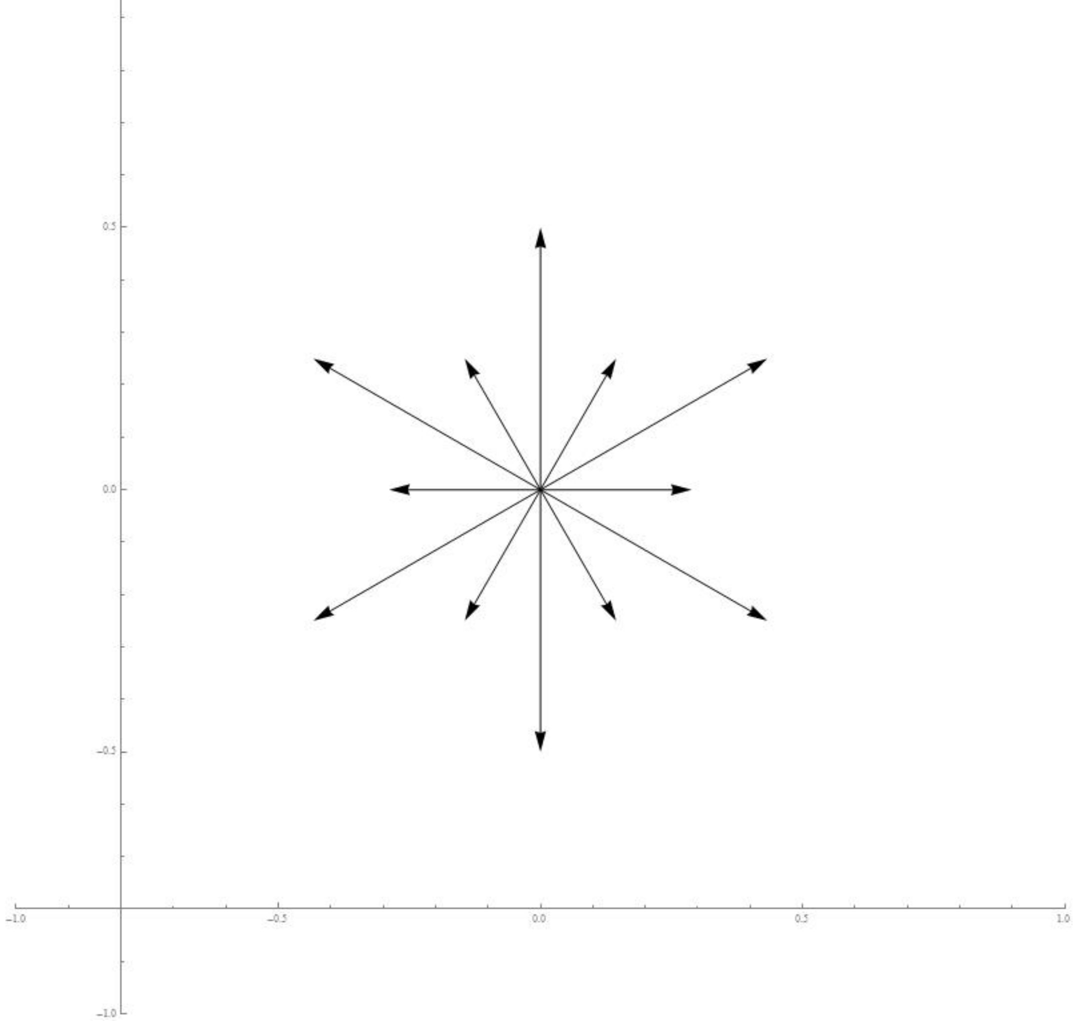
$$\mathfrak{o}(4)\ =\ \mathfrak{o}^{+}(4)\ \oplus\ \mathfrak{o}^{-}(4). \tag{14.17}$$

14.2 Example: $\mathfrak{g}_2$

There is a single Lie algebra of rank 2: $\mathfrak{sl}_2 \approx \mathfrak{sp}_2 \approx \mathfrak{o}_3$.

There are four semisimple Lie algebras of rank 2: $\mathfrak{sl}_2 \times \mathfrak{sl}_2 \approx \mathfrak{o}_4$, $\mathfrak{sl}_3$, $\mathfrak{sp}_4 \approx \mathfrak{o}_5$, and $\mathfrak{g}_2$.

The only other simple Lie algebra that has a maximal toral subalgebra of dimension less than three is $\mathfrak{g}_2$. This Lie algebra can be defined as the Lie algebra of derivations on the purely imaginary octonions. It has the following root system:


1.0
0.5
0.0
-0.5
-1.0
-1.0
-0.5
0.0
0.5
1.0

14.3 Example: $\mathfrak{g}_2$

The smallest representation as a matrix group is by 7×7 matrices. We have $\mathfrak{g}_2 \subset \mathfrak{o}(7)$. A basis for a maximal toral subalgebra can be taken to be

$$n_1 = \frac{\sqrt{-3}}{12}\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & -2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 \end{pmatrix} \tag{14.18}$$

$$n_2 = \frac{\sqrt{-1}}{4}\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix} \tag{14.19}$$

The rest of the Lie algebra is given by the following matrices

$$x_1 = \begin{pmatrix} 0 & 0 & 0 & -2 & 2i & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -i & -1 \\ 0 & 0 & 0 & 0 & 0 & 1 & -i \\ 2 & 0 & 0 & 0 & 0 & 0 & 0 \\ -2i & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & i & -1 & 0 & 0 & 0 & 0 \\ 0 & 1 & i & 0 & 0 & 0 & 0 \end{pmatrix} \quad y_1 = \begin{pmatrix} 0 & 0 & 0 & -2 & -2i & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & i & -1 \\ 0 & 0 & 0 & 0 & 0 & 1 & i \\ 2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 2i & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -i & -1 & 0 & 0 & 0 & 0 \\ 0 & 1 & -i & 0 & 0 & 0 & 0 \end{pmatrix} \tag{14.20}$$

$$x_2 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -i & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & -i & 0 & 0 \\ 0 & i & -1 & 0 & 0 & 0 & 0 \\ 0 & 1 & i & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \quad y_2 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & i & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & i & 0 & 0 \\ 0 & -i & -1 & 0 & 0 & 0 & 0 \\ 0 & 1 & -i & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \tag{14.21}$$

$$x_3 = \begin{pmatrix} 0 & 2 & 2i & 0 & 0 & 0 & 0 \\ -2 & 0 & 0 & 0 & 0 & 0 & 0 \\ -2i & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & i & -1 \\ 0 & 0 & 0 & 0 & 0 & 1 & i \\ 0 & 0 & 0 & -i & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & -i & 0 & 0 \end{pmatrix} \quad y_3 = \begin{pmatrix} 0 & 2 & -2i & 0 & 0 & 0 & 0 \\ -2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 2i & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -i & -1 \\ 0 & 0 & 0 & 0 & 0 & 1 & -i \\ 0 & 0 & 0 & i & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & i & 0 & 0 \end{pmatrix} \tag{14.22}$$

$$x_4 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -i & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & i \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & i & -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & -i & 0 & 0 & 0 & 0 \end{pmatrix} \quad y_4 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & i & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & -i \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -i & -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & i & 0 & 0 & 0 & 0 \end{pmatrix} \tag{14.23}$$

$$x_5 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & -2i & 2 \\ 0 & 0 & 0 & -1 & -i & 0 & 0 \\ 0 & 0 & 0 & -i & 1 & 0 & 0 \\ 0 & 1 & i & 0 & 0 & 0 & 0 \\ 0 & i & -1 & 0 & 0 & 0 & 0 \\ 2i & 0 & 0 & 0 & 0 & 0 & 0 \\ -2 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \quad y_5 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & -2i & 2 \\ 0 & 0 & 0 & -1 & -i & 0 & 0 \\ 0 & 0 & 0 & -i & 1 & 0 & 0 \\ 0 & 1 & i & 0 & 0 & 0 & 0 \\ 0 & i & -1 & 0 & 0 & 0 & 0 \\ 2i & 0 & 0 & 0 & 0 & 0 & 0 \\ -2 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \tag{14.24}$$

$$x_6 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -i & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & i \\ 0 & 0 & 0 & i & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & -i & 0 & 0 \end{pmatrix} \quad y_6 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & i & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & -i \\ 0 & 0 & 0 & -i & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & i & 0 & 0 \end{pmatrix} \tag{14.25}$$

The matrices representing the adjoints of n_1 and n_2 are diagonal, and we have

$$\begin{aligned} ad\, n_1 &= diag\left(0, 0, \frac{\sqrt{3}}{6}, -\frac{\sqrt{3}}{6}, \frac{\sqrt{3}}{4}, -\frac{\sqrt{3}}{4}, \frac{\sqrt{3}}{12}, -\frac{\sqrt{3}}{12}, 0, 0, -\frac{\sqrt{3}}{12}, \frac{\sqrt{3}}{12}, -\frac{\sqrt{3}}{4}, \frac{\sqrt{3}}{4}\right) \\ ad\, n_2 &= diag\left(0, 0, 0, 0, \frac{1}{4}, -\frac{1}{4}, \frac{1}{4}, -\frac{1}{4}, \frac{1}{2}, -\frac{1}{2}, \frac{1}{4}, -\frac{1}{4}, \frac{1}{4}, -\frac{1}{4}\right) \end{aligned} \tag{14.26}$$

in the $n_1, n_2, x_1, y_1, x_2, y_2, x_3, y_3, x_4, y_4, x_5, y_5, x_6, y_6$ ordered basis. The roots are therefore

$$\begin{aligned}
\alpha_1 &= \left(\frac{\sqrt{3}}{6}, 0\right) \\
\alpha_2 &= \left(\frac{\sqrt{3}}{4}, \frac{1}{4}\right) \\
\alpha_3 &= \left(\frac{\sqrt{3}}{12}, \frac{1}{4}\right) \\
\alpha_4 &= \left(0, \frac{1}{2}\right) \\
\alpha_5 &= \left(-\frac{\sqrt{3}}{12}, \frac{1}{4}\right) \\
\alpha_6 &= \left(-\frac{\sqrt{3}}{4}, \frac{1}{4}\right)
\end{aligned} \tag{14.27}$$

and their negatives. Root lengths are therefore of length $\frac{1}{2}$ and $\frac{\sqrt{3}}{6}$.

Chapter 15

Root System Axiomatics

Nov 1, 2012

In this lecture we examine root systems from an axiomatic point of view.

15.1 Reflections

If $v \in \mathbb{R}^n$, then it determines a hyperplane, denoted P_v, through the origin. Reflection about this hyperplane, denoted σ_v, is just

$$\sigma_v(x) \;=\; x \,-\, \frac{2\,(x,\,v)}{(v,\,v)}\,v. \tag{15.1}$$

We make the general definition

$$\langle\, w,\; v\,\rangle \;\triangleq\; \frac{2\,(w,\,v)}{(v,\,v)} \tag{15.2}$$

meaning

$$\sigma_v(x) \;=\; x \,-\, \langle x,\, v\rangle\; v \tag{15.3}$$

Lemma 15.1.1 *Let $\Phi \subset \mathbb{R}^n$ be a finite set which spans $\mathbb{R}^n$, and also has the property that $v \in \Phi$ implies that σ_v leaves Φ invariant. Assume $\sigma \in GL(\mathbb{R}^n)$ leaves Φ invariant, fixes a hyperplane $P \subset \mathbb{R}^n$ in the pointwise sense, and sends some vector $v \in \mathbb{R}^n$ to its negative. Then $\sigma = \sigma_v$ and $P = P_v$.*

Pf. Set $\tau = \sigma \circ \sigma_v$, so obviously $\tau : \Phi \to \Phi$. Note that $\tau : \mathbb{R}v \to \mathbb{R}v$ acts as the identity. We can identify $\mathbb{R}^n \approx P \oplus \mathbb{R}v$ so $w \in \mathbb{R}^n$ is $w = w_P + w_v$, $w_P \in P$ and $w_v \in \mathbb{R}v$. We have $\sigma_v(w_P + w_v) = w_P - \frac{2(w_P,v)}{(v,v)}v - w_v$ so $\tau(w_P) = w_P + \frac{2(w_P,v)}{(v,v)}v + w_v$. Thus τ acts on $\mathbb{R}^n/\mathbb{R}v \approx P$, and acts as the identity. This means τ has only unity eigenvalues, so its minimal polynomial divides $(T-1)^n$. If $w \in \Phi$ then $\{w, \tau(w), \ldots, \tau^k(w)\}$ cannot all be independent if $k \geq Card(\Phi)$. Thus there is some number l so that τ^l acts as the identity on Φ and therefore on $\mathbb{R}^n$ (l will be the l.c.m. of the various k for the various $w \in \Phi$). Thus also the minimal polynomial of τ divides $T^l - 1$. However $gcd\left(T^k - 1, (T-1)^n\right) = T - 1$, so the minimal polynomial is $T - 1$. Thus $\tau = Id$. □

15.2 Root System Axiomatics

We say a subset Φ of $E = \mathbb{R}^n$ (with its Euclidean metric) is a *root system* provided

R1) Φ is finite, spans E, and does not contain 0

R2) If $\alpha \in \Phi$, then $c\alpha \in \Phi$ if only if $c = \pm 1$

R3) If $\alpha \in \Phi$, then σ_α leaves Φ invariant

R4) If $\alpha, \beta \in \Phi$, then $\langle \beta, \alpha \rangle \in \mathbb{Z}$

The dimension of $\mathbb{R}^n$ is called the *rank* of the root system.

Given a root system $\Phi \in E$, denote by $\mathcal{W} \subset GL(E)$ the group of transformations generated by σ_α for all $\alpha \in \Phi$. This is called the *Weyl group* for the root system.

Lemma 15.2.1 *If $\sigma \in GL(E)$ leaves Φ invariant, then $\sigma\sigma_\alpha\sigma^{-1} = \sigma_{\sigma(\alpha)}$ and $\langle \beta, \alpha \rangle = \langle \sigma(\beta), \sigma(\alpha) \rangle$ for all $\alpha, \beta \in \Phi$.*

Pf. There is some $\gamma \in \Phi$ so that $\sigma^{-1}(\beta) = \gamma$. Thus

$$\begin{aligned} \sigma\sigma_\alpha\sigma^{-1}(\beta) &= \sigma\sigma_\alpha(\gamma) \\ &= \sigma\left(\gamma - \langle \gamma, \alpha \rangle \alpha\right) \end{aligned} \tag{15.4}$$

and since $\gamma - \langle \gamma, \alpha \rangle \alpha$ is a root, so is $\sigma(\gamma - \langle \gamma, \alpha \rangle \alpha)$. Thus $\sigma\sigma_\alpha\sigma^{-1}$ leaves Φ invariant. Also, $\sigma\sigma_\alpha\sigma^{-1}$ sends $\sigma(\alpha)$ to $-\sigma(\alpha)$, and fixes the plane $\sigma(P_\alpha)$. By the lemma we have $\sigma\sigma_\alpha\sigma^{-1} = \sigma_{\sigma(\alpha)}$.

For the second assertion,

$$\begin{aligned} \sigma\left(\beta - \langle \sigma(\beta), \sigma(\alpha) \rangle \alpha\right) &= \sigma(\beta) - \langle \sigma(\beta), \sigma(\alpha) \rangle \sigma(\alpha) \\ &= \sigma_{\sigma(\alpha)}(\sigma(\beta)) \\ &= \sigma\sigma_\alpha\sigma^{-1}\sigma(\beta) \\ &= \sigma\sigma_\alpha(\beta) \\ &= \sigma\left(\beta - \langle \beta, \alpha \rangle \alpha\right). \end{aligned} \tag{15.5}$$

Taking σ^{-1} we get the assertion. $\square$

If $\alpha \in \Phi$ define its "dual" $\hat{\alpha}$ to be

$$\hat{\alpha} \;=\; \frac{2\alpha}{(\alpha,\,\alpha)} \tag{15.6}$$

and define $\hat{\Phi} = \{\hat{\alpha} \,|\, \alpha \in \Phi\}$. This is also a root system, and the Weyl groups of Φ and $\hat{\Phi}$ are canonically isomorphic. In the Lie algebra situation, $\hat{\alpha}$ is the metric dual (under the Killing form) to h_α.

Proposition 15.2.2 *Assume $\alpha, \beta \in \Phi$. We have $\langle \alpha, \beta \rangle \langle \beta, \alpha \rangle \geq 0$. The only possibilities for $\langle \alpha, \beta \rangle$ are $0, \pm 1. \pm 2, \pm 3$. If θ is the angle between α and β, then (up to sign) θ can only take on the values $0, \frac{\pi}{6}, \frac{\pi}{4}, \frac{\pi}{3}, \frac{\pi}{2}, \frac{2\pi}{3}, \frac{3\pi}{4}, \frac{5\pi}{6}$. Finally, one of $\langle \alpha, \beta \rangle$, $\langle \beta, \alpha \rangle$ is ± 1.*

Pf. Since $\cos\theta = \frac{(\alpha,\beta)}{|\alpha||\beta|}$ so

$$\langle \alpha,\, \beta \rangle \langle \beta, \alpha \rangle \;=\; 4\cos^2(\theta). \tag{15.7}$$

Since $0 \leq 4\cos^2\theta \leq 4$, and because $4\cos^2\theta = 4$ only when $\theta = 0, \pi$, the first two assertions hold. Obviously $\cos\theta \;=\; 0, \pm 1, \pm\frac{1}{\sqrt{2}}, \pm\frac{\sqrt{3}}{2}$, so the third assertion also holds. The final assertion is obvious from the fact that $\theta \neq \pi$. $\square$

Proposition 15.2.3 *Let $\alpha, \beta \in \Phi$ be non-proportional roots. If $(\alpha, \beta) > 0$ (the angle is strictly acute) then $\alpha - \beta$ is a root. If $(\alpha, \beta) < 0$ (the angle is strictly obtuse) then $\alpha + \beta$ is a root.*

Pf. The second assertion follows from the first, by replacing α by $-\alpha$. Both $\langle \alpha, \beta \rangle, \langle \beta, \alpha \rangle$ are positive, so by the previous proposition, one of them equals 1. If $\langle \alpha, \beta \rangle = 1$ then

$$\sigma_\beta(\alpha) \;=\; \alpha \;-\; \langle \alpha, \beta \rangle\, \beta \;=\; \alpha - \beta \tag{15.8}$$

is in Φ. If $\langle \beta, \alpha \rangle = 1$ then

$$\sigma_\alpha(\beta) \;=\; \beta \;-\; \langle \beta, \alpha \rangle\, \alpha \;=\; \beta - \alpha \tag{15.9}$$

is in Φ, so also $\alpha - \beta \in \Phi$. $\square$

The set of all roots of the form $\beta + i\alpha$ for $i \in \mathbb{Z}$ is called the *α-string through β*.

Proposition 15.2.4 *Facts about strings:*

- *The α-string through β is unbroken.*

- *σ_α reverses all α-strings.*
- *Strings have length at most* 4.

Pf. Let $q, r \in \mathbb{Z}^{\geq 0}$ be integers so $\beta - r\beta \in \Phi$ but $\beta - r'\alpha \notin \Phi$ for any $r' > r$, and so $\beta + q\alpha \in \Phi$ by $\beta + q'\alpha \notin \Phi$ for any $q' > q$.

Now assume p, s can be found with $-r \leq p < p+1 \leq s-1 < s \leq q$ so that $\beta + p\alpha \in \Phi$ and $\beta + s\alpha \in \Phi$, by $\beta + p'\alpha \notin \Phi$ for all $p+1 \leq p' \leq s-1$. Then according to the lemma, we have both $(\alpha, \beta) + p|\alpha|^2 = (\alpha, \beta + p\alpha) > 0$ and $(\alpha, \beta) + s|\alpha|^2 = (\alpha, \beta + s\alpha) < 0$. Thus $p < -\frac{(\alpha,\beta)}{|\alpha|^2} < s$ which is impossible by assumption.

Because σ_α reverses the string, we have

$$\sigma_\alpha(\beta + q\alpha) \; = \; \beta - r\alpha \tag{15.10}$$

so that

$$\begin{aligned} \beta \, - \, \langle \beta, \alpha \rangle \, \alpha \, - \, q\alpha \; &= \; \beta - r\alpha \\ - \, \langle \beta, \alpha \rangle \, \alpha \; &= \; (q - r)\alpha \end{aligned} \tag{15.11}$$

Therefore $|q - r|$ is at most 4. □

15.3 Bases and the Weyl group

We have seen that new roots can be constructed from old, via strings. The question naturally arises, can we find some minimal set of roots that generates all the others?

A subset $\triangle \subset \Phi$ is called a *base* if

B1) $\triangle$ is a basis of E

B2) Each root β can be written as a sum $\beta = \sum c_\alpha \alpha$ where $\alpha \in \triangle$ and $k_\alpha \in \mathbb{Z}$ where all k_α are either all positive or all negative (or zero).

If a base for Φ exists, we can say a root is positive or negative if its coefficients are all positive or negative. We can also define the *height* of a root by $ht\beta = \sum k_\alpha$ where $\beta = \sum k_\alpha \alpha$. Of course if a root system has more than one base, a root might have different heights relative to these bases. However because any given base is a basis, a root has a well-defined height relative to a given base. Finally we have an ordering of roots. Given roots β, α we have

$$\alpha \; > \; \beta \tag{15.12}$$

provided $\alpha - \beta$ is a root and is positive. Obviously this is only a partial ordering.

Lemma 15.3.1 *If $\triangle$ is a base of Φ, then $(\alpha, \beta) \leq 0$ for $\alpha \neq \beta$ for all $\alpha, \beta \in \triangle$, and $\alpha - \beta$ is not a root.*

Pf. Easy. $\square$

Let $\gamma \in E$ be any vector. We call γ *regular* if it does not lie in P_α for any $\alpha \in \Phi$. Define $\Phi^+(\gamma)$ to be

$$\Phi^+(\gamma) \ = \ \left\{ \alpha \in \Phi \ \middle| \ (\alpha, \gamma) \ > \ 0 \right\}. \tag{15.13}$$

Clearly if γ is regular, then $\Phi = \Phi^+(\gamma) \cup -\Phi^+(\gamma)$. An element $\alpha \in \Phi$ is called *indecomposable* provided $\alpha = \beta_1 + \beta_2$ implies that either β_1 or β_2 is not in $\Phi^+(\gamma)$.

Theorem 15.3.2 *Let $\gamma \in E$ be regular. Then the set $\triangle(\gamma)$ of indecomposable roots in $\Phi^+(\gamma)$ is a base. Further, every base $\triangle$ is a $\triangle(\gamma)$ for some regular γ.*

Pf. Next time. $\square$

Chapter 16

Simple Roots

October 30, 2012

16.1 Existence of a base

Recall that a subset $\triangle \subset \Phi$ is called a base if

B1) $\triangle$ is a basis of E

B2) Each root β can be written as a sum $\beta = \sum c_\alpha \alpha$ where $\alpha \in \triangle$ and $k_\alpha \in \mathbb{Z}$ where all k_α are either positive or all negative (or zero).

If $\triangle$ is a base for Φ then we call any element $\alpha \in \triangle$ a *simple root.*

Theorem 16.1.1 *Let $\gamma \in E$ be regular. Then the set $\triangle(\gamma)$ of indecomposable roots in $\Phi^+(\gamma)$ is a base. Further, every base $\triangle$ is a $\triangle(\gamma)$ for some regular γ.*

Pf. Step I. *Every root in $\Phi^+(\gamma)$ is a $\mathbb{Z}$-linear combination of elements in $\triangle(\gamma)$*. If not, there is some $\alpha \in \Phi^+(\gamma)$ for which this fails; we can assume such α is chosen so (α, γ) is a small as possible. Clearly α is decomposable: $\alpha = \beta_1 + \beta_2$ for some $\beta_1, \beta_2 \in \Phi^+(\gamma)$. Then

$$(\alpha,\, \gamma) \;=\; (\alpha,\, \beta_1) \,+\, (\alpha,\, \beta_2)\,, \tag{16.1}$$

and since both terms on the right are positive, β_1 and β_2 must be $\mathbb{Z}$-linear combination of elements in $\triangle(\gamma)$ (or else the minimality of (α, γ) is violated). However then α is a $\mathbb{Z}$-linear combination of elements in $\triangle(\gamma)$, a contradiction.

Step II. *If $\alpha, \beta \in \triangle(\gamma)$, then $(\alpha, \beta) \leq 0$.* If $(\alpha, \beta) > 0$ then $\alpha - \beta$ is a root, so either $\alpha - \beta$ or $\beta - \alpha$ is in $\Phi^+(\gamma)$. In the first case $\alpha = (\alpha - \beta) + \beta$ and in the second $\beta = (\beta - \alpha) + \alpha$, so at least one of α, β is a positive sum of elements in $\Phi^+(\gamma)$, which is impossible.

Step III. *$\triangle(\gamma)$ is a linearly independent set.* Suppose $0 = \sum r_\alpha \alpha$ for elements $\alpha \in \triangle(\gamma)$. Splitting this up, we have

$$\sum p_\alpha \alpha \;=\; \sum n_\beta \beta \tag{16.2}$$

where each $p_\alpha, n_\beta > 0$. Because $(\alpha, \beta) \leq 0$ we have

$$|\sum_\alpha p_\alpha \alpha|^2 \;=\; \Big(\sum p_\alpha \alpha,\, \sum n_\beta \beta\Big) \;=\; \sum_{\alpha,\beta} p_\alpha n_\beta (\alpha, \beta) \;\leq\; 0 \tag{16.3}$$

so that necessarily $\sum_\alpha p_\alpha \alpha = 0$. But then $0 = \sum_\alpha p_\alpha\,(\alpha, \gamma)$, and since $(\alpha, \gamma) > 0$, we have $p_\alpha = 0$. Similarly $n_\beta = 0$.

Step IV. *$\triangle(\gamma)$ is a base of $\Phi^+(\gamma)$.* Since $\triangle(\gamma)$ is linearly independent and spans $\Phi^+(\gamma)$ over the positive integers, it spans E. Clearly also every element in $\Phi(\gamma)$ is either a completely positive or completely negative linear combination of elements in $\triangle(\gamma)$.

Step V. *Each base $\triangle$ is of the form $\triangle(\gamma)$ for some $\gamma \in E$.* The positive half-spaces H_α associated to the elements $\alpha \in \triangle(\gamma)$ have non-empty intersection (this is an exercize). Choose any vector γ in $\bigcap_{\alpha \in \triangle} H_\alpha$. □

16.2 Weyl Chambers

The set of hyperplanes P_α where $\alpha \in \Phi$ divide E into open cones, each of which is called a *Weyl Chamber*. Each regular γ lie in exactly one Weyl chamber, which we call $\mathfrak{C}(\gamma)$. Further, from the theorem it is pretty clear that the bases are in one-one correspondence; we call the chamber corresponding to $\triangle$ the *fundamental Weyl chamber relative to* $\triangle$, denoted $\mathfrak{C}(\triangle)$. The chamber is characterized as follows:

$$\mathfrak{C}(\triangle) \;=\; \big\{\, \beta \in E \;\big|\; (\beta,\, \alpha) > 0 \text{ for all } \alpha \in \triangle \,\big\}. \tag{16.4}$$

Finally, if $\sigma \in \mathcal{W}$ is an element of the Weyl group, we have

$$\begin{aligned} \sigma\left(\mathfrak{C}(\gamma)\right) \;&=\; \mathfrak{C}(\sigma(\gamma)) \\ \sigma\left(\triangle(\gamma)\right) \;&=\; \triangle(\sigma(\gamma)) \end{aligned} \tag{16.5}$$

16.3 Simple Roots

Lemma 16.3.1 *If α is a positive but non-simple root, then $\alpha - \beta$ is a positive root for some $\beta \in \triangle$.*

Pf. If $(\alpha, \beta) \leq 0$ for all $\beta \in \triangle$ then $\triangle \cup \{\beta\}$ would be a linearly independent set, by the proof of Step III above. This is impossible because $\triangle$ is a basis of E. Thus $(\alpha, \beta) > 0$ for some $\beta \in \triangle$, meaning $\alpha - \beta$ is a root. Writing $\alpha = \sum k_\gamma \gamma$ for elements $\gamma \in \triangle$, we have that the expansion of $\alpha - \beta$ has at least one positive coefficient, so the expansion is therefore completely positive. □

Corollary 16.3.2 *Each $\beta \in \Phi^+$ can be written in the form $\beta = \alpha_1 + \cdots + \alpha_k$ with each $\alpha_i \in \triangle$ (and not necessarily distinct) so that each partial sum $\alpha_1 + \cdots + \alpha_i$ is a root.*

Pf. The lemma, along with induction on $ht\,\beta$. □

Lemma 16.3.3 *If $\alpha \in \triangle$, then σ_α permutes all positive roots except α itself.*

Lemma 16.3.4 *Defining the vector δ by $\delta = \frac{1}{2}\sum_{\beta \in \Phi^+} \beta$, we have $\sigma_\alpha(\delta) = \delta - \alpha$ for all $\alpha \in \triangle$.*

Pf. Obvious from the lemma. □

Lemma 16.3.5 *Let $\{\alpha_i\}_{i=1}^t \in \triangle$ (not necessarily distinct), and write $\sigma_i = \sigma_{\alpha_i}$. If $\sigma_1 \ldots \sigma_{t-1}(\alpha_t)$ is negative, then for some index s with $1 \leq s \leq t-1$, we have*

$$\sigma_1 \ldots \sigma_t = \sigma_1 \ldots \sigma_{s-1}\sigma_{s+1} \ldots \sigma_{t-1} \tag{16.6}$$

Pf. For $0 \leq i \leq t-2$, set $\beta_i = \sigma_{i+1} \ldots \sigma_{t-1}(\alpha_t)$, and $\beta_{t-1} = \alpha_t$. We have $\beta_1 < 0$ and $\beta_{t-1} > 0$. Thus there is a smallest k so that $\beta_k > 0$. Therefore $\sigma_k(\beta_k) < 0$, so by the lemma we must have $\beta_k = \alpha_k$ Thus

$$\begin{aligned} \sigma_k &= \sigma_{\alpha_k} \\ &= \sigma_{\sigma_k \ldots \sigma_{t-1}\alpha_t} \\ &= \sigma_{k+1} \ldots \sigma_{t-1}\sigma_{\alpha_t}\sigma_{t-1} \ldots \sigma_{k+1} \end{aligned} \tag{16.7}$$

which proves the lemma. □

Corollary 16.3.6 *Let $\sigma = \sigma_1 \ldots \sigma_t$ be an expression for $\sigma \in \mathcal{W}$ in terms of reflections in simple roots, with t as small as possible. Then $\sigma(\alpha_t) < 0$.*

Pf. Apply the lemma. □

Chapter 17

The Weyl Group

November 1, 2012

17.1 Information on Weyl groups

Theorem 17.1.1 *Let $\triangle$ be a base of Φ*

a) If $\gamma \in E$ is regular, there exists some $\sigma \in \mathcal{W}$ so that $(\sigma(\gamma), \alpha) > 0$ for all $\alpha \in \triangle$; in particular $\mathcal{W}$ acts transitively on Weyl chambers

b) If $\triangle'$ is any other base, there is an element $\sigma' \in \mathcal{W}$ so that $\sigma'(\triangle') = \triangle$; in particular $\mathcal{W}$ acts transitively on bases

c) If α is any root, there exists some $\sigma \in \mathcal{W}$ so that $\sigma(\alpha) \in \triangle$

d) $\mathcal{W}$ is generated by the σ_α for $\alpha \in \triangle$

e) If $\sigma(\triangle) = \triangle$ for some $\sigma \in \mathcal{W}$, then $\sigma = 1$; in particular $\mathcal{W}$ acts simply transitively on the bases.

Pf. For the time being we'll make a distinction between $\mathcal{W}'$, the group generated by reflection in elements in $\triangle$, and the full Weyl group $\mathcal{W}$.

a) Set $\delta = \frac{1}{2}\sum_{\alpha\in\Phi^+} \alpha$. Choose σ so that $\big(\sigma^{-1}(\delta), \gamma\big)$ is as small as possible. But then

$$\big(\sigma^{-1}(\delta), \gamma\big) \;\geq\; \big(\sigma^{-1}(\sigma_\alpha\delta),\, \gamma\big) \;=\; (\delta\,-\,\alpha,\, \sigma\gamma) \;=\; \big(\sigma^{-1}\delta,\, \gamma\big)\,-\,(\alpha,\, \sigma\gamma) \qquad (17.1)$$

Equality exists only when γ is not regular, which is assumed not to be the case. Thus $(\sigma\gamma, \alpha) > 0$ as promised.

b) We have seen that $\mathcal{W}'$ transitively permutes the Weyl chambers. Since every base determine a chamber, there is an element $\sigma \in \mathcal{W}'$ that takes the chamber $\mathcal{C}$ (corresponding to $\triangle$) to the chamber $\mathcal{C}'$ (corresponding to $\triangle'$). By the indecomposability of $\triangle$ and $\triangle'$, they must now be the same (up to ordering).

c) Find a vector γ so that $\alpha \perp \gamma$ but so that γ is not perpendicular to any other root. Slightly perturbing γ, we can make (α, γ) smaller than any other positive (β, γ). Clearly then α is indecomposable.

d) Given any root β, by (*b*) and (*c*) we know there is a transformation $\sigma \in \mathcal{W}'$ so that $\sigma(\beta) \in \triangle$. Then

$$\sigma_\beta \;=\; \sigma^{-1}\sigma_{\sigma^{-1}\beta}\sigma \tag{17.2}$$

is an element of $\mathcal{W}'$. Thus any generator of $\mathcal{W}$ is an element of $\mathcal{W}'$, so that $\mathcal{W} = \mathcal{W}'$.

e) Any element $\sigma = \sigma_{\alpha_1} \dots \sigma_{\alpha_t}$ can be written minimally. By a previous lemma, σ takes at least one positive element (namely α_t) to a negative element, and therefore does not act on $\triangle$ itself.

□

According to the theorem, given any map $\sigma \in \mathcal{W}$ we can write

$$\sigma \;=\; \sigma_{\alpha_1} \dots \sigma_{\alpha_t} \tag{17.3}$$

where $\alpha_i \in \triangle$. If t is minimal, we say t is the *length* of the map σ relative to the base $\triangle$, denoted $l(\sigma)$. We also define $l(1) = 0$. Define $n(\sigma)$ to be the number of positive roots $\beta > 0$ for which $\sigma(\beta) < 0$.

Lemma 17.1.2 *Given any $\sigma \in \mathcal{W}$, we have $n(\sigma) = l(\sigma)$.*

Pf. This is an induction on $l(\sigma)$. Clearly it is true for $l(\sigma) = 0$. Assume the theorem holds for all τ with $l(\tau) \leq t-1$, and let σ be so that $l(\sigma) = t$. Then if $\sigma = \sigma_{\alpha_1} \dots \sigma_{\alpha_t}$ is a minimal expression for σ, we have that σ sends α_t to $-\alpha_t$. Then $\sigma' = \sigma\sigma_{\alpha_t} = \sigma_{\alpha_1} \dots \sigma_{\alpha_{t-1}}$ (which is also minimal) sends α_t to α_t, but since σ_{α_t} sends all positive roots besides α_t to positive roots, we have $n(\sigma') = n(\sigma) - 1$. Since clearly also $l(\sigma') = l(\sigma) - 1$, the theorem is proved by induction. □

Lemma 17.1.3 *Assume λ, μ are vectors in the closure of $\mathfrak{C}(\triangle)$, and assume some $\sigma \in \mathcal{W}$ has $\sigma\lambda = \mu$. Then σ is a product of simple reflections that fix λ; in particular $\lambda = \mu$.*

Pf. Since σ sends at least one simple root, say α, to a negative root, we have

$$0 \geq (\sigma\alpha,\, \mu) \;=\; \left(\alpha,\, \sigma^{-1}\mu\right) \;=\; (\alpha,\, \lambda) \;\geq\; 0 \tag{17.4}$$

Therefore equality holds, so $\sigma_\alpha\lambda = \lambda$. But then $\sigma'\lambda = \sigma\sigma_\alpha\lambda = \sigma\lambda = \mu$. However $l(\sigma') = l(\sigma) - 1$, so we can apply induction on $l(\sigma)$ to obtain the result. □

17.2 Irreducible root systems

A root system Φ is called irreducible if it cannot be partitioned into non-empty subsets, so that the elements of either subset is perpendicular to all vectors in the other.

Lemma 17.2.1 *A root system Φ is irreducible if and only if every base $\triangle$ is irreducible.*

Pf. If a base $\triangle$ is reducible so that $\triangle = \triangle' \cup \triangle''$, then the Weyl group, which is generated by that base, is also reducible. This is due to the fact that if $(\alpha, \beta) = 0$ then $\sigma_\alpha \sigma_\beta = \sigma_\beta \sigma_\alpha$, so $\mathcal{W} = \mathcal{W}' \times \mathcal{W}''$. Every root is conjugate to a simple root, but since $\mathcal{W}'$, $\mathcal{W}''$ fix, respectively, the (orthogonal) subspaces spanned by $\triangle'$, $\triangle''$, any root that is conjugate to an element of $\triangle'$ is not conjugate to an element of $\triangle''$, and vice-versa. Thus since the subspaces $span\triangle'$ and $span\triangle''$ are fixed under $\mathcal{W}$, every root is either in one or the other, so therefore the root system is decomposable. The converse is even easier. □

Lemma 17.2.2 *Let Φ be an irreducible root system. Relative to the partial ordering $<$ on Φ^+ there is a unique maximal root β (in particular $ht\,\beta > ht\,\alpha$ for all $\alpha \in \Phi$ and $(\beta, \alpha) \geq 0$ for all $\alpha \in \Phi^+$).*

Pf. Choose a β so that β is maximal among all roots that it is comparable to. We first prove that it is comparable to all simple roots. If not, there is some $\alpha \in \triangle$ so that $\beta - \alpha$ is not a root. Thus $(\beta, \alpha) \leq 0$. But if equality holds, then α is orthogonal to all simple roots that comprise β, so that $\triangle$ is partitioned orthogonally, an impossibility. Since $(\beta, \alpha) < 0$, we have that $\beta + \alpha$ is a root, and $\beta + \alpha > \beta$, an impossibility.

Since β is comparable to all simple roots, we can see it is comparable to all positive roots. Specifically, if $\beta - \alpha$ is not a root then $(\beta, \alpha) \leq 0$, but equality cannot hold because both are positive linear combinations of base roots and β involves all base roots, so because $(\beta, \alpha) < 0$ then $\beta + \alpha$ is a root and is comparable to β, contradicting the maximality of β. □

Lemma 17.2.3 *Let Φ be irreducible. Then $\mathcal{W}$ acts irreducibly on E. In particular the $\mathcal{W}$-orbit of any $\alpha \in \Phi$ spans E.*

Pf. If $\mathcal{W}$ does not act irreducibly, so $E' \subset E$ is a proper subspace preserved by $\mathcal{W}$, then the orthogonal compliment E'' also has an action of $\mathcal{W}$. By reducibility, clearly either $\alpha \in E'$ or else $E' \subset P_\alpha$ and similarly for E''. As a consequence all roots lie in either E' or E'', contradicting the irreducibility of Φ. □

Lemma 17.2.4 *Let Φ be irreducible. Then at most two root lengths occur in Φ, and all roots of a given length are conjugate under $\mathcal{W}$.*

Pf. Provided $(\alpha,\beta)\neq 0$, the only possible ratios of their length-squares are $\frac{1}{3},\frac{1}{2},1,2,3$. Further, for any α, its orbit under $\mathcal{W}$ contains a vector α' so that $(\alpha',\beta)>0$. This proved the first assertion, since the existence of three root lengths would imply a ratio of $\frac{2}{3}$.

Let α,β be roots of the same root length. We may assume $(\alpha,\beta)>0$. If they are distinct, then one of $\langle\alpha,\beta\rangle$, $\langle\beta,\alpha\rangle$ is ± 1, and therefore $\langle\alpha,\beta\rangle=\langle\beta,\alpha\rangle=+1$. Then

$$(\sigma_\alpha\sigma_\beta\sigma_\alpha)\,(\beta) \;=\; (\sigma_\alpha\sigma_\beta)\,(\beta-\alpha) \;=\; \sigma_\alpha\,(-\alpha) \;=\; \alpha \tag{17.5}$$

□

Lemma 17.2.5 *Let Φ be irreducible, and have two root lengths. Then the maximal root is long.*

Pf. Let β be the maximal root. Because β is comparable to all positive roots, $(\beta,\alpha)\geq 0$ for all $\alpha\in\Phi^+$, and equality cannot hold, so β is in the fundamental Weyl chamber. Given any α, it is in some chamber, and since the Weyl group transitively permutes chamber, we can assume α is in the fundamental chamber. Then $\beta-\alpha$ is a positive root, and $(\gamma,\beta-\alpha)\geq 0$ for any γ in the closure of the fundamental chamber. Then $|\beta|^2-(\beta,\alpha)\geq 0$ and $(\alpha,\beta)-|\alpha|^2\geq 0$, so that $|\beta|^2>(\alpha,\beta)\geq|\alpha|^2$. □

Chapter 18

Classification of Root Systems, I

November 6, 2012

Notes from this day are missing.

Chapter 19

Classification of Root Systems II

November 8, 2012

Notes from this day are missing.

Chapter 20

Lie Algebras and the Proof of the Isomorphism Theorem

November 20, 2012

Notes from this day are missing.

Chapter 21

Examples

November 22, 2012

Notes from this day are missing.

Chapter 22

Cartan and Engel Subalgebras (Book: ch 15)

November 27, 2012

22.1 Inner Automorphisms

Notes from this section are missing.

22.2 Engel Subalgebras

Arbitrary (finite dimensional) algebra L over $\mathbb{C}$.

Let $x \in L$. Generalized eigenspace decomposition: $L_a(ad\,x)$ characterized by the maximality of $(ad\,x - a)^m\big|_{L_a(ad\,x)} = 0$. Then

$$L = \bigoplus_{a \in \mathbb{C}} L_a(ad\,x) \tag{22.1}$$

(sum taken over non-trivial $L_a(ad\,x)$) is the generalized eigenspace decomposition.

Easy facts: $[L_a(ad\,x), L_b(ad\,x)] \subseteq L_{a+b}(ad\,x)$. Thus $L_a(ad\,x)$ for $a \neq 0$ consists of nilpotent endomorphisms, and $L_0(ad\,x)$ is a subalgebra containing x. The $L_0(ad\,x)$ are called *Engel subalgebras*.

Lemma 22.2.1 *Let $K \subseteq L$ be a subalgebra. Let $z \in K$ have (1) $L_0(ad\, z)$ minimal among all $L_0(adx)$, $x \in K$ and (2) $K \subseteq L_0(ad\, z)$. Then $L_0(ad\, z) \subseteq L_0(ad\, x)$, all $x \in K$.*

Pf. Set $K_0 = L_0(ad\, z)$, so $z \in K_0 \subseteq K$. Pick $x \in K$ and consider $ad(z + cx)$, for varying $c \in \mathbb{C}$. Because $z + cx \in K \subseteq K_0$ we have that $z_c x$ stabilizes the algebra K_0, and therefore passes to an endomorphism of L/K_0. The characteristic polynomial of $z + cx$ therefore splits into

$$g(T, c)\, f(T, c) \tag{22.2}$$

(put, say, $T = z + cz$) where $g(T, c)$ is the char. poly. on L/K_0 and f is the char. poly on K_0. We have

$$f(T, c) \;=\; T^r \;+\; f_1(c)T^{r-1} \;+\; \ldots \;+\; f_r(c) \tag{22.3}$$

$$g(T, c) \;=\; T^{n-r} \;+\; g_1(c)T^{n-r-1} \;+\; \ldots,\; g_{n-r}(c) \tag{22.4}$$

and each f_i, g_i is a polynomial of degree i or less.

Let's examine $g(T, c)$. For $c = 0$, the eigenvalue 0 occurs only on K_0 so $ad\, z : L/K_0$ is an isomorphism. Thus $g(T, 0)$ has a constant term, so $g_{n-r}(c)$ is not precisely 0. Pick $r + 1$ many values $c_1, \ldots, c_{r+1}$ where $g_{n-r}(c_i) \neq 0$. For each of these c_j, we have that $ad(z + cx)$ is an isomorphism on L/K_0, so $L_0(ad(z + c_j x)) \subseteq K_0$. This forces $L_0(ad(z + c_j x)) = K_0$ by the minimality of K_0. Therefore $f(T, c_j) = T^r$. In particular $f_r(c)$ is a polynomial of degree r or less with $r + 1$ or more zeros. It is therefore the zero polynomial. Therefore $ad(z + cx)|_{K_0}$ has only zero eigenvalues, so $L_0(ad(z + cx)) \subseteq K_0$ for any $c \in \mathbb{C}$, $x \in K$. Pick $x - z$ and $c = 1$. □

Lemma 22.2.2 *Assume K is a subalgebra of L that contains an Engel subalgebra. Then K is self-normalizing. In particular all Engel subalgebras are self-normalizing.*

Pf. If K contains an Engel subalgebra, there is some $x \in K$ so that $L_0(ad\, x) \subseteq K$. We have that $ad\, x : N_L(K) \to N_L(K)$ and since $x \in K$ stabilizes K, that $ad\, x : N_L(K)/K \to L/K$ is an isomorphism because $L_0(ad\, x) \subseteq K$. But also $ad\, x : N_L(K) \to K$. Thus $N_L(K) = K$. □

22.3 Cartan Subalgebras

A Cartan subalgebra (CSA) is a nilpotent, self-normalizing subalgebra. In the semisimple case, we have root space decomposition

$$L = H \oplus \bigoplus_{\alpha \in \Phi} L_\alpha \tag{22.5}$$

where H is abelian (hence nilpotent) and $N_L(H) = H$. Thus CSA's exist.

Theorem 22.3.1 *A subalgebra $H \subset L$ is a CSA if and only if H is a minimal Engel subalgebra.*

Pf. First assume $H = L_0(ad\,x)$ and properly contains no other Engel subalgebra. We have $N_L(H) = H$ by the previous lemma. Setting $H = K$, the first lemma gives that $H = L_0(ad\,z) \subseteq L_0(ad\,x)$ for all $x \in H$. Thus each $ad\,x$ is nilpotent on H, so H is *ad*-nilpotent, therefore nilpotent.

Second, assume H is a CSA. By nilpotency, $H \subseteq L_0(ad\,x)$, all $x \in L$. Set $H' = L_0(ad\,z)$ where $z \in H$ is chosen so H' is minimal (and $H \subset H'$). By the lemma, $L_0(ad\,x)$ contains H', all $x \in H$. Therefore H acts on H'/H via nilpotent transformations, so there is a vector $y + H \in H'/H$ which is a common zero eigenvector. Thus H is normal in $H \oplus \mathbb{C}y$. □

Corollary 22.3.2 *If L is semisimple, then H is a CSA if and only if it is a maximal toral subalgebra.*

Pf. We have already noted that a maximal toral is a CSA. Now assume H is a CSA, so $H = L_0(ad\,x)$ for some x. Note that $L_0(ad\,x_s) \subseteq L_0(ad\,x)$, so we can assume x is abstractly semisimple. Further for x semisimple, $L_0(ad\,x) = C_L(x)$. Clearly $C_L(x)$ contains a minimal toral subalgebra, which is a CSA and therefore minimal Engel. But because $H = C_L(x)$ is a CSA and therefore minimal Engel, it must coincide with the maximal toral. □

Note that just because $x = x_s$ is semisimple doesn't mean $L_0(ad\,x) = C_L(X)$ is a CSA. Those semisimple x for which $C_L(x)$ is a CSA are called *regular semisimple* elements.

22.4 Functoral properties

Lemma 22.4.1 *If $\varphi : L \to L'$ is an epimorpism, it maps CSAs to CSAs.*

Pf. Let H be a CSA. Obviously $\varphi(H)$ is nilpotent. If $x' \in N_{L'}(\varphi(H))$ there is some $x \in L$ so $x' = \varphi(x)$ and $x \notin H$ Then $\varphi([x, N]) \subseteq \varphi(N)$ so $[x, N] \subseteq N$, and N is normal in $N \oplus \mathbb{C}x$, contradicting self-normalization. □

Lemma 22.4.2 *Let $\varphi : L \to L'$ be an epimorphism, and let H' be a CSA of L'. Setting $K = \varphi^{-1}(N')$, and CSA of K is a CSA of L.*

Pf. Let H be a CSA of K. Then $\varphi(H)$ is a CSA of L' lying in H', so $\varphi(H) = H'$. Then if $x \in L$ has $[x, H] \subseteq H$, we have that $\varphi(x)$ normalizes H' so $\varphi(x) \in H'$. Thus $x \in H \oplus ker(\varphi) \subseteq K$. But H is a CSA of K, so in fact $x \in H$. □

22.5 More terminology

We call $x \in L$ strongly nilpotent if $x \in L_a(ad\, y)$ for some $y \in L$. Note that x strongly nilpotent implies x is nilpotent.

Set $\mathcal{N}(L)$ to be the subspace of all strongly nilpotent elements, and let $\mathcal{E}(L)$ be the subgroup of $Int(L)$ generated by $\mathcal{N}(L)$. Notice $\mathcal{E}(L)$ is normal in $Aut(L)$.

A Borel subalgebra is a maximal solvable subalgebra. For example, if

$$L = H \oplus \bigoplus_{\alpha \in \Phi} L_\alpha \tag{22.6}$$

is a semisimple algebra, then if $\triangle \subseteq \Phi$ is a choice of base, then setting

$$B(\triangle) \;=\; H \oplus \bigoplus_{\alpha > 0} L_\alpha \tag{22.7}$$

$$N(\triangle) \;=\; \bigoplus_{\alpha > 0} L_\alpha \tag{22.8}$$

we have $N(\triangle) = [B(\triangle, B(\triangle)]$ is nilpotent, so $B(\triangle)$ is solvable. One easily sees it is maximal (by stability under H it must include some L_α with $\alpha < 0$), so it is a Borel subalgebra.

Facts:

Borel subalgebras are self-normalizing.

Borel subalgebras of L are in 1-1 correspondance to Borel subalgebras of the semisimple algebra $L/Rad(L)$.

If L is semisimple and maximal toral H is chosen, then choice of base $\triangle$ is a choice of a Borel subalgebra. All such choices are conjugate under $\mathcal{E}(L)$ (therefore under $Int(L)$).

Chapter 23

Examples

November 29, 2012

Notes from this day are missing.

Chapter 24

Conjugacy Theorems

December 4, 2012

Notes from this day are missing.

Part II

Spring 2013

Coming soon!