Moduli Spaces of critical Riemanian Metrics with $L^{\frac{n}{2}}$ norm curvature bounds

Xiuxiong Chen and Brian Weber

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Abstract

We consider the moduli space of the extremal Kähler metrics on compact manifolds. We show that under the conditions of two-sided total volume bounds, $L^{\frac{n}{2}}$-norm bounds on Rm, and Sobolev constant bounds, this Moduli space can be compactified by including (reduced) orbifolds with finitely many singularities. Most of our results go through for certain other classes of critical Riemannian metrics.

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1 Introduction

A Kähler metric is called extremal if the complex gradient of its scalar curvature is a holomorphic vector field. This includes the more famous Kähler Einstein metrics and constant scalar curvature Kähler (cscK) metrics as special cases, though one would like to understand the structure of extremal metrics as well. In this note, we propose to study the weak compactness of extremal Kähler metrics in a bounded family of Kähler classes together with bounds on the $L^{\frac{n}{2}}$ norm of Riemannian curvature and on the Sobolev constants. The extremal Kähler metric equation is naturally a 6th order equation on Kähler potential, and its compactness properties are difficult to study directly. We essentially decompose the extremal condition into three inter-related second order equations as below:

$$\triangle \mathrm{Rm} = \nabla^2 \mathrm{Ric} + \mathrm{Rm} * \mathrm{Rm} \tag{1}$$
$$\triangle \mathrm{Ric} = \mathrm{Ric} * \mathrm{Rm} + \nabla X \tag{2}$$
$$\triangle X = \mathrm{Rm} * X. \tag{3}$$

The " $*$ " stands for tensor contraction between two multi-index tensors (more elaboration on this below) and X is a vector field related to the critical Riemannian metric[1]. A large class of critical metrics satisfy these three coupled equations, for instance CSC Bach-flat metrics, harmonic curvature metrics, and

[1]For the extremal Kähler metrics, X is the complex gradient vector field of the scalar curvature function.

Einstein metrics, all of which have been studied before. Below we show that another class of metrics, the extremal Kähler metrics, also satisfy these equations.

More specifically we study the weak compactness of the space $\mathcal{M} = \mathcal{M}(n, C_S, \Lambda, \nu, \delta)$ of critical metrics (where X is non-trivial) that satisfy[2]

i) energies are bounded: $\int_M |\operatorname{Rm}|^{\frac{n}{2}} \le \Lambda$

ii) volumes are bounded from below: $\operatorname{Vol} M \ge \nu$, and

iii) diameters are bounded from above: $\operatorname{dist}_M(x, y) \le \delta$, all $x, y \in M$.

iv) the Sobolev constant C_g on (M, g) has a uniform bound, $C_g \le C_S$.

The Sobolev inequality referred to here controls the embedding $W^{1,2} \hookrightarrow L^{\frac{2n}{n-2}}$, and usually takes the form

$$\left(\int \phi^{2\gamma}\right)^{1/\gamma} \le C_g \int |\nabla\phi|^2 + \frac{A}{(\operatorname{Vol} M)^{2/n}} \int \phi^2,$$

where $\gamma = \frac{n}{n-2}$ and $\phi \in C^1$. In fact one often takes $max\,(C_g, A)$ to be the Sobolev constant. We require the simplified form of the inequality,

$$\left(\int \phi^{2\gamma}\right)^{1/\gamma} \quad \le \quad C_g \int |\nabla\phi|^2.$$

If one assumes $\operatorname{Vol}(\operatorname{supp}\phi)$ is smaller than $A/2\sqrt{\nu}$, then we can use it in this form. In Section 2.5 we show how sometimes A and C_S are automatically controlled.

In this paper, we study the weak compactness in all dimensions of our "critical metrics", which satisfy conditions (*i*)-(*iv*) above. There is a substantial body of prior compactness results which we build on. The case of CSC Bach-flat, harmonic curvature, and CSC Kähler metrics was considered in [TV1], [TV2]. Recent work of Anderson's [And2], [And3] elaborates on this theme. These works in turn can be traced back to work of M. Anderson [And1], G. Tian [Tia1], and Bando-Kasue-Nakajima [BKN] on the moduli space of Einstein metrics on four dimensional manifolds with L^2 norm curvature bound. These in turn were natural extensions of earlier work of J. Cheeger [Che] and later M. Gromov [Gro], which explored geometric and topological control on manifolds with various pointwise bounds on curvature. Readers are encouraged to read [CT] for more references.

[2]For complex surfaces, the only assumption is the Sobolev constant. The others are either *a priori* or can be derived from *a priori* constraints. Moreover, there is a large open set of Kähler classes where also the Sobolev constant is *a priori* bounded for the extremal representatives, c.f. Section 2.5.

To analyze the inter-play of the three coupled equations, one must obtain some *a priori* bounds on the $X = \nabla^{(1,0)} R$ (R indicates scalar curvature throughout). Without using the assumption of Sobolev constant bound, we derive an L^2 norm bound on X and ∇X in all dimensions (cf. Lemma 2.1). This is important for both geometrical and analytical reasons. Analytically, this $W^{2,2}$ bound on R, together with a bound on $\| \operatorname{Rm} \|_{\frac{n}{2}}$, serves as our starting point for a weak compactness argument on the moduli space of extremal metrics. Geometrically, the L^∞ bound on scalar curvature (likewise the L^2 bound on X) is a consequence of the scalar curvature map (from complex structure) being a moment map (if interpreted correctly). It is more difficult to to understand what $\int_M |\nabla \bar{\nabla} R|^2$ represents geometrically however. A natural question is whether all $W^{k,2}$ norms of the scalar curvature function are *a priori* bounded.

Perhaps the main technical theorem we prove is the usual ϵ-regularity

Theorem 1.1 *(cf. Theorem 4.1) Assume g is a critical metric on a Riemannian manifold. When $a > \frac{n}{2}$ and $q \in \{0, 1, \dots\}$, there exists $\epsilon_0 = \epsilon_0(C_S, a, q, n)$ and $C = C(C_S, a, q, n)$ so that*

$$\int_{B(o,r)} |\operatorname{Rm}|^{\frac{n}{2}} \leq \epsilon_0$$

implies

$$\left(\int_{B(o,r/2)} |\nabla^q X|^a \right)^{\frac{1}{a}} \leq C r^{-q-3+\frac{n}{a}} \left(\int_{B(o,r)} |R|^{\frac{n}{2}} \right)^{\frac{2}{n}} \tag{4}$$

$$\left(\int_{B(o,r/2)} |\nabla^q \operatorname{Ric}|^a \right)^{\frac{1}{a}} \leq C r^{-q-2+\frac{n}{a}} \left(\int_{B(o,r)} |\operatorname{Ric}|^{\frac{n}{2}} \right)^{\frac{2}{n}} \tag{5}$$

$$\left(\int_{B(o,r/2)} |\nabla^q \operatorname{Rm}|^a \right)^{\frac{1}{a}} \leq C r^{-q-2+\frac{n}{a}} \left(\int_{B(o,r)} |\operatorname{Rm}|^{\frac{n}{2}} \right)^{\frac{2}{n}}. \tag{6}$$

This is obtained by interactive use of the three equations. From a purely technical point of view, the case $n > 4$ is more complicated than the case of $n = 4$ (in the smooth case at least). For $n > 4$, we derive all three estimates simultaneously using an induction argument (see appendix). The proof is lengthy and technical and we hope it can be shortened in the future.

The main theorems we prove are:

Theorem 1.2 *(cf. Theorem 4.6) Assume g is a critical metric on a Riemannian manifold. Then there exists an $\epsilon_0 = \epsilon_0(C_S, n, p)$ and $C = C(C_S, n, p)$ so* *that* $\int_{B_r} |\operatorname{Rm}|^{\frac{n}{2}} \leq \epsilon_0$ *implies*

$$\sup_{B(o,r/2)} |\nabla^p \operatorname{Rm}| \; \leq \; C r^{-p-2} \left(\int_{B(o,r)} |\operatorname{Rm}|^{\frac{n}{2}} \right)^{\frac{2}{n}}.$$

And, specializing to the case of extremal Kähler manifolds,

Theorem 1.3 (Orbifold compactness) *(cf. Theorem 5.6) Assume* $\{(M_\alpha, J_\alpha, \omega_\alpha)\}$ *is a family of compact extremal Kähler manifolds that satisfy conditions (i) - (iv). Then a subsequence converges in the Gromov-Hausdorff topology to a (reduced) compact extremal Kähler orbifold. Further, there is a bound* $C_1 = C_1(\Lambda, C_S, n)$ *on the number of singularities, and a bound* $C_2 = C_2(C_S, n)$ *on the order of any orbifold group.*

If the family does not consist of extremal metrics but their metrics satisfy the elliptic system (1), (2), (3) and conditions (*i*)-(*iv*), this theorem still holds, except that the singularities are only of orbifold type C^0, and are not necessarily reduced (meaning a tangent cone could be a one-point union of standard cones over various $\mathbb{S}^3/\Gamma$). There is a variety of classes of metrics that satisfy (1), (2), and (3), for instance the CSC Bach-flat and harmonic curvature metrics considered in [TV1], where in fact $X = 0$.

A nontrivial step in proving orbifold compactness is to prove a uniform upper bound on the local volume ratio. If there is a *pointwise* lower bound on Ricci curvature, then this upper bound is automatic from the Bishop-Gromov comparison theorem. We do not assume such curvature lower bounds, so we prove that volume growth is uniformly bounded by generalizing a result of Tian-Viaclovsky's [TV1], [TV2] to cover our class of critical metrics in all dimensions. In [TV1] Tian-Viaclovsky proved that complete manifolds with bounded energy, bounded Sobolev constant, and quadratic curvature decay $|\operatorname{Rm}| = o(r^{-2})$ have finitely many ALE ends and therefore a global upper bound on volume growth. This represented a major advance; previous results had required a nearly unusable strengthening of the curvature decay condition. In [TV2] they use this to prove uniform volume ratio bounds on compact manifolds with certain critical metrics, without pointwise bounds on Ricci curvature.

Recall that a specified structure, say a differential manifold structure or a vector bundle structure, is said to exist on an orbifold if it exists at all manifold points and, after lifting, can be completed on any local orbifold cover. In the 4-dimensional case, in the absence of additional rigidity, the analytic methods presently known are only strong enough to show that the orbifold metric is continuous (see [And2]).

Showing that the completion of the orbifold metric (on a smooth orbifold cover) is C^∞ requires a way to remove apparent point-singularities. In higher dimensions, powerful analytic techniques, developed originally to remove singularities in Yang-Mills instantons, suffice to remove the singularities in our case as well (e.g. Lemma 3.4, Proposition 3.5). The critical case is real dimension 4, where these analytic techniques fail. Here one needs the geometry itself to provide additional rigidity. We find this rigidity, in the case of extremal Kähler metrics, in a partially improved Kato inequality (Lemma 4.14), which we take advantage of using Uhlenbeck's broken Hodge gauge technique ([Uhl], [Tia1], [TV1]).

In [TV1] an improved Kato inequality was shown to hold for 4-dimensional CSC Riemannian metrics with $\delta W^+ = 0$ in the (sharp) form

$$|\nabla|E||^2 \;\le\; \frac{2}{3}|\nabla E|^2,$$

where E is the trace-free Ricci tensor. This is actually a consequence of the theory of Kato constants developed in [Bra] and [CGH]. This is sufficient for applications to Kähler geometry, where for instance constant scalar curvature implies that W^+ is constant. We are able to use a direct argument to partially recover an improved inequality. Specifically, we get

$$2|\nabla|\nabla E||^2 \;\le\; \frac{1}{4}|\nabla\nabla E|^2 \;+\; |\overline{\nabla}\nabla E|^2. \tag{7}$$

This does not quite give sufficient control on the Hessian of E; see Proposition 4.14 and its use in Proposition 4.15. As a result, the removable singularity theorem becomes correspondingly more complicated, and we need to utilize Uhlenbeck's technique in slightly different manner. Our Kato inequality represents a mild extension of the existing theory, the main difference being that we are forced to consider a $U(n)$, not $SO(n)$ decomposition of tensors. As usual, the improved Kato inequality yields an improved elliptic inequality, which (via Uhlenbeck's method) allows one to improve the behavior of $|\,\mathrm{Rm}\,|$ at singularities or at infinity.

Remark. In an interesting recent work [CS], a corresponding precompactness result for Kähler-Ricci solitons was derived with the additional assumption of pointwise Ricci curvature bounds. These bounds on Ricci curvature in [CS] can be removed as in our case. The details will be found in a forthcoming paper [Web2].

Organization. In section 3 we consider the steps necessary for attaining moduli space compactness under our assumptions, and establish the analytic lemmas needed to overcome these difficulties. In section 4.1 we state the necessary estimates and outline the Moser iteration argument needed for weak compactness. In section 5 we give the weak compactness argument; we also give the proof of the volume growth upper bound, and state a gap theorem for ALE extremal metrics. We also present our adaptation of the argument for attaining uniform volume growth bounds. Some details will be omitted from various arguments, as they are nearly identical to those found elsewhere.

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2 A quick introduction to Kähler geometry

2.1 Setup of notations

Let M be an n-dimensional compact Kähler manifold. A Kähler metric can be given by its Kähler form ω on M. In local coordinates $z^1, \cdots, z^n$, this ω has the form

$$\omega = \sqrt{-1} \sum_{i,j=1}^{n} g_{i\bar{j}} d\, z^i \wedge d\, \bar{z}^j > 0,$$

where $\{g_{i\bar{j}}\}$ is a positive definite Hermitian matrix function. The Kähler condition requires that ω is a closed positive (1,1)-form, or in other words, that

$$\frac{\partial g_{i\bar{k}}}{\partial z^j} = \frac{\partial g_{j\bar{k}}}{\partial z^i} \qquad \text{and} \qquad \frac{\partial g_{k\bar{i}}}{\partial \bar{z}^j} = \frac{\partial g_{k\bar{j}}}{\partial \bar{z}^i} \qquad \forall\, i, j, k = 1, 2, \cdots, n.$$

The Hermitian metric corresponding to ω is given by

$$\sqrt{-1} \sum_{1}^{n} g_{\alpha\bar{\beta}}\, d\, z^{\alpha} \otimes d\, \bar{z}^{\beta}.$$

For simplicity we will often denote by ω the corresponding Kähler metric. The Kähler class of (M, ω) is the cohomology class $[\omega]$ in $H^2(M, \mathcal{R})$. The curvature tensor is

$$\mathrm{Rm}_{i\bar{j}k\bar{l}} = -\frac{\partial^2 g_{i\bar{j}}}{\partial z^k \partial \bar{z}^l} + \sum_{p,q=1}^{n} g^{p\bar{q}} \frac{\partial g_{i\bar{q}}}{\partial z^k} \frac{\partial g_{p\bar{j}}}{\partial \bar{z}^l}, \qquad \forall\, i, j, k, l = 1, 2, \cdots n.$$

The Ricci curvature of ω is locally given by

$$\mathrm{Ric}_{i\bar{j}} = -\frac{\partial^2 \log \det(g_{k\bar{l}})}{\partial z^i \partial \bar{z}^j},$$

so its Ricci curvature form is

$$\mathrm{Ric} \;=\; \sqrt{-1}\sum_{i,j=1}^{n} \mathrm{Ric}_{i\bar{j}}\, d\, z^i \wedge d\, \bar{z}^j \;=\; -\sqrt{-1}\partial\overline{\partial} \log\, \det(g_{k\bar{l}}).$$

It is a real, closed (1,1)-form.

2.2 Historic background and motivation

In 1982, E. Calabi [Cal1] proposed to study the critical metrics of the so called "Calabi energy" in each Kähler class:

$$Ca(\omega) = \int_M (R - \underline{R})^2\, \omega^n.$$

The critical metrics for this functional (the so-called *extremal* Kähler metrics) satisfy the following equation

$$R_{,\bar{\alpha}\bar{\beta}} = 0, \qquad \forall\, \alpha,\ \beta = 1, 2, \cdots, n.$$

In other words, the extremal Kähler metrics are just those where the complex gradient field of the scalar curvature functions is a holomorphic vector field. This class includes the Kähler-Einstein metrics, and more generally the constant scalar curvature (cscK) metrics. The famous conjecture of Calabi states that if the first Chern class (C_1) has a definite sign, then there is a Kähler Einstein metric in the canonical Kähler class. The celebrated work of T. Aubin [Aub] ($C_1 < 0$), S. T. Yau [Yau] ($C_1 < 0$ and $C_1 = 0$) and G. Tian [Tia1] ($C_1 > 0$ for complex surfaces) settles the Calabi conjecture in these respective cases. The remaining case ($C_1 > 0$ and dimension > 2) is much more complicated ([Tia2]). In the 1980s, when he introduced the notion of extremal Kähler metrics, E. Calabi initially expected that there would exists an extremal Kähler metric in each Kähler class. This conjecture of Calabi is known to be false as stated since there are certain algebraic obstructions to the existence of extremal Kähler metrics ([Lev]). We know our list of obstructions is incomplete however, as Tian [Tia2] constructed a example where the known obstructions vanish but there is no cscK metric.

There is relatively little progress on the general existence problem using PDE methods, although there is very active research in utilizing the special symmetric structure of underlying Kähler manifold as well as in deploying subtle implicit function methods (cf. [Cal1] [LS1] [LS2] [ACGT] [AP] [APS] [Fine] and references therein) to construct (or prove the existence of) extremal Kähler metrics. The present work is a movement in this direction using geometric methods.

2.3 Derivation of some useful formulas

First we show how to derive the elliptic system (1), (2), and (3). We note that the first equation holds for any Riemannian manifold, though the derivation in the Kähler case is simpler. We compute in unitary frames

$$\begin{aligned}
\mathrm{Rm}_{i\bar{j}k\bar{l},m\bar{m}} &= \mathrm{Rm}_{i\bar{j}m\bar{l},k\bar{m}} \\
&= \mathrm{Rm}_{i\bar{j}m\bar{l},\bar{m}k} + \mathrm{Rm}_{k\bar{m}i\bar{s}}\mathrm{Rm}_{s\bar{j}m\bar{l}} - \mathrm{Rm}_{k\bar{m}s\bar{j}}\mathrm{Rm}_{i\bar{s}m\bar{l}} \\
&\qquad + \mathrm{Rm}_{k\bar{m}m\bar{s}}\mathrm{Rm}_{s\bar{j}s\bar{l}} - \mathrm{Rm}_{k\bar{m}s\bar{l}}\mathrm{Rm}_{i\bar{j}m\bar{s}} \\
&= \mathrm{Ric}_{i\bar{j},\bar{l}k} + \mathrm{Rm}_{k\bar{m}i\bar{s}}\mathrm{Rm}_{s\bar{j}m\bar{l}} - \mathrm{Rm}_{k\bar{m}s\bar{j}}\mathrm{Rm}_{i\bar{s}m\bar{l}} \\
&\qquad + \mathrm{Ric}_{k\bar{s}}\mathrm{Rm}_{s\bar{j}s\bar{l}} - \mathrm{Rm}_{k\bar{m}s\bar{l}}\mathrm{Rm}_{i\bar{j}m\bar{s}}.
\end{aligned}$$

When the exact form of the expression is not important we will denote a linear combination of traces of tensor products of S and T simply by $S * T$. Using this of abbreviation, we write

$$\triangle\,\mathrm{Rm} \quad = \quad \mathrm{Rm} * \mathrm{Rm} + \nabla\overline{\nabla}\,\mathrm{Ric}\,.$$

Next we work with the Ricci tensor, and note that a simplification of $\triangle\,\mathrm{Ric}$ is possible in the Kähler case because we are allowed additional permutations of indices.

$$\begin{aligned}
\mathrm{Ric}_{i\bar{j},m\bar{m}} &= \mathrm{Ric}_{m\bar{j},i\bar{m}} \\
&= \mathrm{Ric}_{m\bar{j},\bar{m}i} + \mathrm{Ric}_{i\bar{s}}\mathrm{Ric}_{s\bar{j}} - \mathrm{Rm}_{i\bar{m}s\bar{j}}\mathrm{Ric}_{m\bar{s}} \\
&= R_{,\bar{j}i} + \mathrm{Ric}_{i\bar{s}}\mathrm{Ric}_{s\bar{j}} - \mathrm{Rm}_{i\bar{m}s\bar{j}}\mathrm{Ric}_{m\bar{s}}.
\end{aligned}$$

The computation for $\mathrm{Ric}_{i\bar{j},\bar{m}m}$ is similar. Using the notation $X = \overline{\nabla}R$, we get

$$\triangle\,\mathrm{Ric} \quad = \quad \mathrm{Rm} * \mathrm{Ric} + \nabla X.$$

In the extremal case we can actually get an elliptic equation for X. Recalling that $\overline{\nabla}X = 0$ for extremals, a commutator formula gives

$$\begin{aligned}
X_{,m\bar{m}} &= R_{,\bar{i}m\bar{m}} = -\mathrm{Ric}_{s\bar{i}}R_{,\bar{s}} \\
\triangle X &= \mathrm{Ric} * X.
\end{aligned}$$

Essentially the same computation gives that

$$\nabla^2 X \quad = \quad \mathrm{Rm} * X.$$

2.4 *A priori* bounds on the extremal vector field

In this section we establish preliminary local estimates for $|X|$ and $|\nabla X|$. It is well known that, given a Kähler manifold and a Kähler class, then the L^∞ norm of its scalar curvature function is *a priori* bounded. Moreover, the extremal vector field X is determined up to conjugation. However, one does not expect that the length of $|X|$ with respect to varying extremal metrics has any kind of bound. We are pleasantly surprised that, without use of the Sobolev inequality, we can directly bound $L^2(X)$. By Fatou's lemma, this result will hold on any manifold-with-singularities that arises as the limit of such manifolds. Extremal Kähler metrics have automatic upper and lower bounds on scalar curvature which depends on the complex structure and Kähler class. Using this fact, we can prove

Proposition 2.1 *Assume M is a compact manifold and that $X = \overline{\nabla} R \triangleq R_{,\bar{i}}\, d\bar{z}^i$ is a holomorphic covector field. Then*

$$\int_M |X|^2 \;\leq\; C \sup|R| \int_M |\operatorname{Ric}|^2.$$

and

$$\int_M |\nabla X|^2 \;\leq\; C \sup|R|^2 \int_M |\operatorname{Ric}|^2.$$

for a constant $C = C(n)$.

<u>*Pf*</u>
We deal with $L^2(|\nabla X|)$ first. We use formula (3) in a more specific form,

$$R_{,\bar{i}j\bar{j}} = R_{,\bar{i}\bar{j}j} + \mathrm{Rm}_{\bar{j}j\bar{i}k} R_{,\bar{k}} = -\mathrm{Ric}_{k\bar{i}} R_{,\bar{k}},$$

and integration by parts. Note that $\nabla X = R_{,\bar{i}j} + R_{,\bar{i}\bar{j}} = R_{,\bar{i}j}$.

$$\begin{aligned}
\int |\nabla X|^2 \;&=\; \int R_{,i\bar{j}} R_{,\bar{i}j} \;=\; -\int R_{,i\bar{j}j} R_{,\bar{i}} \;=\; -\int \mathrm{Ric}_{i\bar{k}} R_{,k} R_{,\bar{i}} \\
&=\; \int \mathrm{Ric}_{i\bar{k},\bar{i}} R_{,k} R \;+\; \int \mathrm{Ric}_{i\bar{k}} R_{,k\bar{i}} R \;=\; \int R_{,\bar{k}} R_{,k} R \;+\; \int \mathrm{Ric}_{i\bar{k}} R_{,k\bar{i}} R \\
&=\; \int |X|^2 R \;+\; \int \langle \mathrm{Ric}, \nabla X\rangle\, R \;\leq\; \int |X|^2 R \;+\; \frac{1}{2}\int |\operatorname{Ric}|^2 R^2 \;+\; \frac{1}{2}\int |\nabla X|^2 \\
\int |\nabla X|^2 \;&\leq\; 2\int |X|^2 R \;+\; \int |\operatorname{Ric}|^2 R^2
\end{aligned}$$

We use

$$\frac{1}{2}\triangle R^2 \;=\; |\nabla R|^2 + R\triangle R \;\geq\; |\nabla R|^2 - \frac{R}{\sqrt{n}} |\operatorname{Hess} R|^2 \;=\; |X|^2 - \frac{R}{\sqrt{n}} |\nabla X|.$$

Then

$$\begin{aligned}
\int |X|^2 &\le \frac{1}{\sqrt{n}}\int R|\nabla X| \;\le\; \left(\frac{1}{n}\int R^2\right)^{\frac{1}{2}}\left(\int |\nabla X|^2\right)^{\frac{1}{2}} \\
&\le \left(\frac{1}{n}\int R^2\right)^{\frac{1}{2}}\left(2\int |X|^2 R + \int |\operatorname{Ric}|^2 R^2\right)^{\frac{1}{2}} \\
&\le \frac{\sup|R|}{n}\int R^2 + \frac{1}{2\sup|R|}\int |X|^2 R + \frac{1}{4\sup|R|}\int |\operatorname{Ric}|^2 R^2 \\
\int |X|^2 &\le \frac{2\sup|R|}{n}\int R^2 + \frac{\sup|R|}{2}\int |\operatorname{Ric}|^2
\end{aligned}$$

Thus

$$\int |X|^2 \;\le\; C\sup|R|\int |\operatorname{Ric}|^2.$$

□

2.5 Uniform Sobolev constant bound

The large scale aim of this research program is to contribute to the understanding the Yau-Tian-Donaldson conjecture and the Calabi conjecture. The most immediate natural application is the special case of complex surfaces with Kähler class in the so-called generalized Tian cone. Let us first define

Definition 2.2 *The Kähler class ω of a complex surface is in Tian's cone if*

$$c_1(M)^2 - \frac{2}{3}\frac{(c_1(M)\cdot[\omega])^2}{[\omega]^2} \;>\; 0.$$

A striking observation ([Tia3], [TV2]) of Tian's is that a positive cscK metric in the Tian cone automatically has a uniform Sobelev constant bound. One can modify this to include the case of extremal Kähler metrics: We say a surface's Kähler class lies in the generalized Tian cone if

$$c_1(M)^2 - \frac{2}{3}\left(\frac{(c_1(M)\cdot[\omega])^2}{[\omega]^2} + \frac{1}{64\pi^2}\|\mathcal{F}\|^2\right) \;>\; 0 \tag{8}$$

Here $\|\mathcal{F}\|$ is the norm of the Calabi-Futaki invariant [Fut] in a Mabuchi-Futaki invariant metric [FM]; see [Chn2] for the definition of this norm. More importantly, extremal metrics in this modified Tian cone sometimes enjoy similar properties. In other words, some extremal Kähler metrics in a bounded region of the modified Tian cone have bounds (i)-(iv) *a priori.*

To make sense of this assertion, use

$$\frac{(C_1\cdot[\omega])^2}{[\omega]\cdot[\omega]} \;=\; \frac{1}{32\pi^2}\frac{1}{\mathrm{Vol}}\left(\int R\right)^2$$

and

$$C_1^2 \;=\; \frac{1}{96\pi^2}\int\left(R^2-12|E|^2\right)\;+\;\frac{1}{48\pi^2}\int R^2,$$

where E indicates the trace-free Ricci tensor. If the representative metric happens to be extremal, it turns out that

$$\|\mathcal{F}\|^2 \;=\; 2\left(\int R^2-\frac{1}{\text{Vol}}\left(\int R\right)^2\right).$$

A glance at the Chern-Gauss-Bonnet formula for χ indicates that $\int\left(R^2-12|E|^2\right)$ is a conformal invariant on 4-manifolds, so when (8) holds, we get a bound on the square of the Yamabe minimizer. It is well-known that the Sobolev constant is bounded in the conformal class of a positive Yamabe minimizer (ref!!), where the constant A in (4) is controlled by the Yamabe constant and $L^\infty(R)$. So assuming (8) and a positive Yamabe constant there is abound on the Sobolev constant. Such a bound holds, for example, on del Pezzo surfaces.

2.6 Future work

Due to LeBrun-Simanca [LS1], it is known that the set of Kähler classes (and bounded complex structures) which admit extremal Kähler metrics is open in the Kähler cone. This suggests that it is possible to pursue the existence of the extremal Kähler metrics using the method of continuity. In a subsequent work, we want to study

Problem 2.3 *Let $\{[\omega_n]\}$ be a sequence of Kähler classes which converges to a limiting Kähler class $[\omega_\infty]$. Suppose that the limiting Kähler class is K stable, and suppose that $\{g_n\}$ is a sequence of extremal Kähler metrics in $\{[\omega_n]\}$ respectively. If the g_i all satisfy conditions (i)-(iv), do we have a smooth limit as $i\to\infty$? In other words, will orbifold singularities fail to develop?*

A special case of this problem, perhaps more natural, is

Problem 2.4 *In complex dimension 2, can we solve problem 2.3? What about in the interior of the generalized Tian's Kähler cone[3]? What happens at the border of this modified Kähler cone? What if we don't assume the limiting class is stable?*

Problem 2.5 *If we remove the assumption of uniform bound on Sobolev constant, does some version of Theorem 1.3 still hold? What if we restrict to surfaces only?*

[3]See Section 2.5 for definition.

In a series of remarkable works [Don1] [Don2] [Don3], S. K. Donaldson initiated the study of the existence of extremal Kähler metrics on toric surfaces; see also [Zhu] for further work in toric varieties. This program might be viewed as parallel to the one described out here. Addressing the problem in full generality would mean tackling one of two essential difficulties: on the one hand the lack of 2-dimensional symmetry in general, and on the other the lack of Sobolev constant control in general. The work of Cheeger-Tian [CT] on 4 dimensional Einstein manifolds may shed some light on this problem.

Problem 2.6 *What can we say about Theorem 1.3 if we assume bounds on the L^2 (instead of $L^{\frac{n}{2}}$) norm of Riemannian curvature?*

In extremal Kähler geometry this is an especially natural question, as the L^2 norm has *a priori* bounds, from which we don't know how to obtain $L^{\frac{n}{2}}$ bounds. There are many important works in this direction by M. Anderson, J. Cheeger, T. Colding, G. Tian, and others. Readers are encouraged to browse [CCT] or [CT] for further details and references.

3 Analytic Lemmas

The results of this section hold for complete manifolds with certain kinds of singular points, what Anderson calls "curvature singularities." Specifically,

Definition 3.1 *Assume M is a length space with a set $S = \bigcup_{j=1}^{N} \{p_j\}$ such that $M - S$ is a smooth Riemannian manifold. If S is the smallest such set, we call it the singular set of M. If for each p_j there is an $\epsilon_j > 0$ and numbers $0 < \underline{v}_j \leq \overline{v}_j$ with the property that $\underline{v}_j r^n \leq \operatorname{Vol} B(p_j, r) \leq \overline{v}_j r^n$ for $0 \leq r < \epsilon_j$, then we call M a manifold-with-singularities, and call the p_j curvature singularities.*

Our goal in this section is to establish the tools we shall need later to establish the pointwise bounds for the Riemannian curvature tensor on manifolds-with-singularities. This provides the first step in both the weak compactness and the removable singularity theorems.

Moser iteration with the elliptic inequality $\triangle u \geq -fu - g$ (roughly the form of (1), (2), and (3)) requires the a priori conditions that $u \in L^2$ and $f, g \in L^p$ for some $p > n/2$. We will have only $p = n/2$ a priori. Essentially by exploiting the *non*linear structure of the system (1), (2), (3), with methods pioneered in [BKN], [Tia1], [And1], we can bootstrap f and g into the needed L^p spaces. The presence of singularities complicates this, the main difficulty being that integration by parts leaves an uncontrollable residue at singularities. The first task is partially recovering integration by parts, which is possible for functions that are differentiable away from the singular set and in $L^{n/(n-1)}_{loc}$ at the singularities.

Remark. The fact that the Sobolev inequality continues to hold for $W^{1,2}$-functions across the singular points, assuming the local upper bound on volume growth, is by now a well known.

Lemma 3.2 (Sobolev inequality for $W^{1,2}$ functions) *Assume the Sobolev inequality $\left(\int_U v^{2\gamma}\right)^{\frac{1}{\gamma}} \leq C_S \int_U |\nabla v|^2$ holds for all domains U with closure $\overline{U}$ compact and disjoint from the singular set, with $\operatorname{Vol} U \leq \frac{1}{2} \operatorname{Vol} M$ if $\operatorname{Vol} M$ is finite, and with $v \in C^1_c(U)$. Then the Sobolev inequality holds for functions $v \in W^{1,2}_0(U)$ even if $\overline{U}$ contains singular points.*

<u>*Pf*</u> See, for instance, the proof of Theorem 5.1 in [BKN] □

Lemma 3.3 (Integration by parts) *Assume X is any vector field with compact support which is smooth outside the singular set. If $|X| \in L^{\frac{n}{n-1}}$ (or just $|X| = O(r^{-(n-1)})$ near singularities) and either $\int (\mathrm{div}(X))_-$ or $\int (\mathrm{div}(X))_+$ is finite, we retain the divergence formula: $\int_M d\left(i_X dVol\right) = 0$.*

<u>*Pf*</u>

Without loss of generality, we assume $(\mathrm{div}(X))_-$ is integrable, and we assume there is only one singularity, at o. For small values of r, let $\phi_r \geq 0$ be a test function with $\phi_r \equiv 1$ outside $B(o, 2r)$, $\phi_r \equiv 0$ inside $B(o, r)$, and $|\nabla \phi_r| \leq 2/r$. Possibly $\int_M \mathrm{div}(X) = +\infty$, but in any case the Dominated Convergence Theorem and Fatou's lemma give

$$\begin{aligned}
\int_M \mathrm{div}(X) &= \int_M (\mathrm{div}(X))_+ - \int_M (\mathrm{div}(X))_- \\
&\leq \lim_{r \to 0^+} \int_M \phi_r (\mathrm{div}(X))_+ - \lim_{r \to 0^+} \int_M \phi_r (\mathrm{div}(X))_- \\
&= \lim_{r \to 0^+} \int_M \phi_r \ \mathrm{div}(X).
\end{aligned}$$

But

$$\begin{aligned}
\left| \int_M \phi_r \ \mathrm{div}(X) \right| &= \left| \int_M \langle X, \nabla \phi_r \rangle \right| \\
&\leq \left(\int_M |\nabla \phi_r|^n \right)^{\frac{1}{n}} \left(\int_{\mathrm{supp}(\nabla \phi_r)} |X|^{\frac{n}{n-1}} \right)^{\frac{n-1}{n}}
\end{aligned}$$

Since $\left(\int_M |\nabla\phi_r|^n\right)^{\frac{1}{n}} \quad \leq \quad \frac{2}{r}\,(\mathrm{Vol}\,B(o,r))^{\frac{1}{n}}$ is finite and $\int_{\mathrm{supp}(\nabla\phi_r)} |X|^{\frac{n}{n-1}}$ is bounded as $r \to 0$, we get that $\lim_{r\to 0}\int_M \phi_r\ \mathrm{div}(X)$ is bounded. Therefore

$$\int_M \mathrm{div}(X) \quad < \quad \infty,$$

which proves that indeed $\int(\mathrm{div}(X))_+ < \infty$. The DCT now gives

$$\int_M \mathrm{div}(X) \quad = \quad \lim_{r\to 0^+}\int_M \phi_r\ \mathrm{div}(X) \ = \ 0.$$

□

We eventually wish to prove that the curvature singularities are "removable," in the sense that the Riemann curvature tensor has uniform pointwise bounds in the neighborhood of any singular point. Thus the singularity may be topologically nontrivial, but its metric structure will be controlled, and in all cases one can prove such a singularity will be a Riemannian orbifold point of regularity at least C^0.

The first step in the removable singularity theorem is establishing that $|\,\mathrm{Rm}\,| \in L^p_{loc}$ for some $p > n/2$. A result of [BKN] is that if $|\,\mathrm{Rm}\,| = O(r^{-2+\alpha})$ for any $\alpha > 0$, one can construct coordinates with $C^{1,1}$ bounds on metric components. In fact, if one can obtain just $C^{1,\alpha}$ coordinates, one has access to harmonic coordinates ([DK]) and a bootstrapping argument can commence, which we give in section 5.

For dimensions 6 and up, we obtain $|\,\mathrm{Rm}\,| \in L^p_{loc}$ using analytic methods first developed in [Sib]. Sibner's original purpose was to prove removable singularity theorems for Yang-Mills instantons, a problem closely related to ours. This method was used again by Cao-Sesum in [CS] to remove singularities on Kähler-Ricci solitons. Sibner's theorem is really only useful in dimension 5 and higher; in the Yang-Mills case other methods were used in dimensions 2, 3, and 4. We use other methods in dimension 4 as well; see section 4.4. The proof below does have some limited applicability in dimensions 3 and 4.

Lemma 3.4 **($u^k \in L^2$ implies $\nabla u^k \in L^2$)** *([Sib]) Assume 2-sided volume growth bounds, Sobolev constant bounds, and $\triangle u \geq -fu$ where $f \in L^{n/2}(B - \{o\})$ and $u \geq 0$. If $k > \frac{1}{2}\frac{n}{n-2}$, then $u^k \in L^2(B - \{o\})$ implies $\nabla u^k \in L^2(B - \{o\})$.*

<u>*Pf*</u>

The idea is to dampen the growth near the singularity while retaining weakly an elliptic inequality. Assume $\frac{1}{2} < q_0 \leq q$, to be chosen later. We set up a test function as follows. Let

$$F(t) = \begin{cases} t^q & \text{if } 0 \leq t \leq l \\ \frac{1}{q_0}\left(ql^{q-q_0}t^{q_0} + (q_0 - q)l^q\right) & \text{if } l \leq t, \end{cases}$$

and set $G(t) = F(t)F'(t)$. We shall need the following easily verified facts:

$$F(t) \leq \frac{q}{q_0} l^{q-q_0} t^{q_0} \tag{9}$$

$$q_0 F(t) \leq tF'(t) \tag{10}$$

$$tF'(t) \leq qF(t) \tag{11}$$

$$G'(t) \geq \frac{2q_0 - 1}{q_0} (F'(t))^2. \tag{12}$$

For a test function ζ,

$$\int \langle \nabla \zeta, \nabla u \rangle \quad \leq \quad \int \zeta u f \tag{13}$$

Choose $\zeta = \eta^2 G(u)$ for our test function. We have to assume $\eta \equiv 0$ across any singularities in order to make integration by parts work. The trick will be to make u disappear. We have

$$\langle \nabla u, \nabla \zeta \rangle \ \geq \ 2\eta F(u) F'(u) \langle \nabla u, \nabla \eta \rangle \ + \ \frac{2q_0 - 1}{q_0} \eta^2 (F'(u))^2 |\nabla u|^2,$$

so combining with (13) and simplifying gives our main inequality:

$$\int 2\eta F(u) \langle \nabla \eta, \nabla F(u) \rangle \ + \ \frac{2q_0 - 1}{q_0} \int \eta^2 |\nabla F(u)|^2 \tag{14}$$

$$\leq \ q \int \eta^2 (F(u))^2 f \tag{15}$$

We deal with the terms one by one. The first term on the left of (15) is easily dispatched with Schwartz:

$$\int 2\eta F(u) \langle \nabla \eta, \nabla F(u) \rangle \quad \geq \quad -\mu \int \eta^2 |\nabla F(u)|^2 \ - \ \frac{1}{\mu} \int |\nabla \eta|^2 |F(u)|. \tag{16}$$

For the term on the right of (15), Hölder and Sobolev give

$$\int \eta^2 (F(u))^2 f \ \leq \ \left(\int f^{\frac{n}{2}} \right)^{\frac{2}{n}} \left(\int \eta^{\frac{2n}{n-2}} (F(u))^{\frac{2n}{n-2}} \right)^{\frac{n-2}{n}}$$

$$\leq \ 2C_S^2 \left(\int f^{\frac{n}{2}} \right)^{\frac{2}{n}} \left(\int |\nabla \eta|^2 F(u)^2 \right) \ + \ 2C_S^2 \left(\int f^{\frac{n}{2}} \right)^{\frac{2}{n}} \left(\int \eta^2 |\nabla F(u)|^2 \right)$$

Putting everything back into (15) and simplifying now gives

$$\left(\frac{2q_0 - 1}{q_0} - \mu - 2qC_S^2 \left(\int f^{\frac{n}{2}} \right)^{\frac{2}{n}} \right) \int \eta^2 |\nabla F(u)|^2$$

$$\leq \ \left(\frac{1}{\mu} + 2qC_S^2 \left(\int f^{\frac{n}{2}} \right)^{\frac{2}{n}} \right) \int |\nabla \eta|^2 |F(u)|^2 \tag{17}$$

After q and q_0 are chosen, we choose μ to be small and choose the cutoff η so that $\left(\int_{\mathrm{supp}(\eta)} f^{\frac{n}{2}}\right)^{\frac{2}{n}}$ is also small. Under these conditions we get that

$$\int \eta^2 |\nabla F(u)|^2 \ \le\ C \int |\nabla \eta|^2 |F(u)|^2, \tag{18}$$

where $C = C(q_0)$, provided that $\|f\|_{L^{\frac{n}{2}}(\mathrm{supp}(\eta))} \le C(C_S, q, q_0)$.

We want to ferret out the contribution at the singularity, so replace η with $\eta\eta_\epsilon$, where now $\eta \equiv 1$ across the singularity, and $\eta_\epsilon \ge 0$ is another cutoff function with $\eta_\epsilon \equiv 1$ outside $B(o, 2\epsilon)$, $\eta_\epsilon \equiv 0$ inside $B(o, \epsilon)$, and $|\nabla \eta_\epsilon| \le 2/\epsilon$. Using $F(v) \le \frac{q}{q_0} l^{q-q_0} u^{q_0}$ and applying Hölder again:

$$\begin{aligned}\int (\eta\eta_\epsilon)^2 |\nabla F(u)|^2 \ &\le\ C\left(\frac{q}{q_0} l^{q-q_0}\right)^2 \left(\int |\nabla \eta_\epsilon|^n\right)^{\frac{2}{n}} \left(\int_{\mathrm{supp}(\nabla\eta_\epsilon)} u^{\frac{2nq_0}{n-2}}\right)^{\frac{n-2}{n}} \\ &\quad + C\int |\nabla\eta|^2 |F(u)|^2.\end{aligned}$$

Now choose $q_0 > \frac{1}{2}$ so $q_0 = k(n-2)/n$ (here we use the hypothesis that $k > \frac{1}{2}\frac{n}{n-2}$). Then $\frac{2nq_0}{n-2} = 2k$ and so $u^{\frac{2nq_0}{n-2}}$ is locally integrable. As $\epsilon \to 0$ we get

$$\left(\int |\nabla \eta_\epsilon|^n\right)^{\frac{2}{n}} \left(\int_{\mathrm{supp}(\nabla\eta_\epsilon)} u^{\frac{2nq_0}{n-2}}\right)^{\frac{n-2}{n}} \to 0,$$

So $\int \eta^2 |\nabla F(u)|^2 \le C \int |\nabla\eta|^2 |F(u)|^2$. Now letting also $l \to \infty$ we finally get

$$\int \eta^2 |\nabla u^q|^2 \ \le\ C \int |\nabla \eta|^2 |u^q|^2. \tag{19}$$

Choosing $q = k$ (so automatically $q > q_0$), we have our result:

$$\nabla u^k \in L^2.$$

□

Proposition 3.5 (L^p-regularity) *Assume $\triangle u \ge -fu - g$, $u \ge 0$ in $B - \{o\}$, with $f, g \in L^{n/2}(B - \{o\})$, and assume 2-sided volume growth bounds at the singular point and a finite Sobolev constant. If $u \in L^q(B - \{o\})$ for some $q > \frac{n}{n-2}$, then $u \in L^p(B - \{o\})$ for all $\infty > p \ge q$. Explicitly, with $a > q > \frac{n}{n-2}$, there exists $\epsilon_0 = \epsilon_0(q, a, C_S)$, $C = C(q, a, C_S, n)$ so that $\int_{B(o,r)} f^{\frac{n}{2}} \le \epsilon_0$ implies*

$$\left(\int_{B(o,r/2)} u^a\right)^{\frac{1}{a}} \ \le\ Cr^{\frac{n}{a}-\frac{n}{q}} \left(\int_{B(o,r)} u^q\right)^{\frac{1}{q}} + Cr^{\frac{n}{a}} \left(\int_{B(o,r)} g^{\frac{n}{2}}\right)^{\frac{2}{n}}. \tag{20}$$

Pf
We must pay special attention to any use of integration by parts; otherwise the argument is standard. Assume $p > 1$. Replace u by $u + \mathcal{C}\|g\|_{L^{\frac{n}{2}}(B(o,r))}$ and f by $f + \frac{1}{\mathcal{C}}\frac{g}{\|g\|_{L^{\frac{n}{2}}(B(o,r))}}$, where $\mathcal{C}$ is some number to be chosen later; it will be roughly $\left(a^2/(a-1)\right)^{n/2}$. Then $\triangle u \geq -fu$. We get

$$\left(\int \left(\eta^2 u^p\right)^{\frac{n}{n-2}}\right)^{\frac{n-2}{n}} \leq 2C_S^2 \int |\nabla \eta|^2 u^p + 2p^2 C_S^2 \int \eta^2 u^{p-2} |\nabla u|^2.$$

The last term reads $C\int \eta^2 |\nabla u^{\frac{p}{2}}|^2$, which we can estimate using (19). This estimate requires that $\left(\int_{B(o,r)} |f|^{\frac{n}{2}}\right)^{\frac{2}{n}}$ be small compared to p and C_S, which, incidentally requires choosing $\mathcal{C}$. We get

$$\left(\int \left(\eta^2 u^p\right)^{\frac{n}{n-2}}\right)^{\frac{n-2}{n}} \leq C \int |\nabla \eta|^2 u^p \tag{21}$$

where $C = C(p, C_S)$. Iterating this inequality will give $u \in L^p$ for all $q \leq p < \infty$.

We carry this out explicitly. With $0 < k < 1$ and an appropriate choice of test functions ϕ, (21) implies

$$\left(\int_{B(o,kr)} u^{p\gamma}\right)^{\frac{1}{\gamma}} \leq Cr^{-2} \int_{B(o,r)} u^p,$$

with $C = C(p, k, C_S)$, and iterating, we get

$$\left(\int_{B(o,k^{i+1}r)} u^{p\gamma^{i+1}}\right)^{\frac{1}{\gamma^{i+1}}} \leq Cr^{\frac{n}{\gamma^{i+1}}-n} \int_{B(o,r)} u^p,$$

with $C = C(p, k, i, C_S)$. Now choose i so $p\gamma^i \leq a < p\gamma^{i+1}$. Then

$$\begin{aligned}
\int_{B(o,k^{i+1}r)} u^a &\leq r^{n-\frac{na}{p}} \left(\int_{B(o,k^{i+1}r)} u^{p\gamma^i}\right)^{\frac{p\gamma^{i+1}-a}{p\gamma^{i+1}-p\gamma^i}} \left(\int_{B(o,k^{i+1}r)} u^{p\gamma^{i+1}}\right)^{\frac{a-p\gamma^i-a}{p\gamma^{i+1}-p\gamma^i}} \\
&\leq Cr^{n-\frac{na}{p}} \left(\int_{B(o,kr)} u^p\right)^{\gamma^i \frac{p\gamma^{i+1}-a}{p\gamma^{i+1}-p\gamma^i}} \left(\int_{B(o,r)} u^p\right)^{\gamma^{i+1} \frac{a-p\gamma^i}{p\gamma^{i+1}-p\gamma^i}} \\
&\leq Cr^{n-\frac{na}{p}} \left(\int_{B(o,r)} u^p\right)^{\frac{a}{p}},
\end{aligned}$$

where $C = C(p,k,a,C_S)$. Now lastly put $u + \mathcal{C}\|g\|_{L^{\frac{n}{2}}(B(o,r))}$ back in for u:

$$\begin{aligned}
\left(\int_{B(o,kr)} u^a\right)^{\frac{1}{a}} &\leq \left(\int_{B(o,r)} (u+\mathcal{C}\|g\|)^a\right)^{\frac{1}{a}} \\
&\leq Cr^{\frac{n}{a}-\frac{n}{p}} \left(\int_{B(o,r)} (u+\mathcal{C}\|g\|)^p\right)^{\frac{1}{p}} \\
&\leq Cr^{\frac{n}{a}-\frac{n}{p}} \left(\int_{B(o,r)} u^p\right)^{\frac{1}{p}} + Cr^{\frac{n}{a}} \left(\int_{B(o,r)} g^{\frac{n}{2}}\right)^{\frac{2}{n}}.
\end{aligned}$$

for $C = C(a,n,k,C_S)$. □

4 Regularity of sectional curvature

4.1 Statement of the curvature estimates

In this section we state our main curvature integral estimates, and actually establish them in the low order case. The method of proof is standard, but establishing the estimates in the possible presence of singularities is more complicated. At smooth points, Propositions 4.2, 4.3, 4.4, and 4.5 give the result for small values of q. The subject of Sections 4.3 and 4.4 is to prove the $q=0$ case at singular points. The rest of the long, unenlightening proof by induction is consigned to the appendix.

Theorem 4.1 *Assume g is an extremal Kähler metric on a Riemannian manifold-with-singularities. When $a > \frac{n}{2}$, and $q \in \{0,1,\dots\}$, there exists $\epsilon_0 = \epsilon_0(C_S,a,q,n)$ and $C = C(C_S,a,q,n)$ so that*

$$\int_{B(o,r)} |\operatorname{Rm}|^{\frac{n}{2}} \leq \epsilon_0$$

implies

$$\left(\int_{B(o,r/2)} |\nabla^q X|^a\right)^{\frac{1}{a}} \leq Cr^{-3-q+\frac{n}{a}} \left(\int_{B(o,r)} |R|^{\frac{n}{2}}\right)^{\frac{2}{n}} \tag{22}$$

$$\left(\int_{B(o,r/2)} |\nabla^q \operatorname{Ric}|^a\right)^{\frac{1}{a}} \leq Cr^{-2-q+\frac{n}{a}} \left(\int_{B(o,r)} |\operatorname{Ric}|^{\frac{n}{2}}\right)^{\frac{2}{n}} \tag{23}$$

$$\left(\int_{B(o,r/2)} |\nabla^q \operatorname{Rm}|^a\right)^{\frac{1}{a}} \leq Cr^{-2-q+\frac{n}{a}} \left(\int_{B(o,r)} |\operatorname{Rm}|^{\frac{n}{2}}\right)^{\frac{2}{n}}. \tag{24}$$

In the presence of singularities estimate (24) holds if $q = 0$ and $n \geq 4$, and estimates (23) and (22) hold if $q = 0, 1$ and $n \geq 6$. In all other cases the estimates hold if $B(o, r)$ consists of manifold points.

We begin the induction argument for Proposition 4.1 at smooth points, for $|\mathrm{Rm}|$, $|\mathrm{Ric}|$ and $|\nabla \mathrm{Ric}|$, and $|X|$ and $|\nabla X|$. It is worth noting that the arguments here work in real dimension $n \geq 3$.

Proposition 4.2 *If $p > \frac{n}{2}$ and $B(o, r)$ consists of smooth points, there exists $\epsilon_0 = \epsilon_0(p, C_S, n)$ and $C = C(p, C_S, n)$ so that $\int_{B(o,r)} |\mathrm{Ric}|^{\frac{n}{2}} \leq \epsilon_0$ implies*

$$\left(\int_{B(o,r/2)} |X|^p\right)^{\frac{1}{p}} \leq C r^{\frac{n}{p}-3} \left(\int_{B(o,r)} |R|^{\frac{n}{2}}\right)^{\frac{2}{n}}$$

<u>*Pf*</u>

This is basically a local version of Proposition 2.1. We obtain the estimates in a series of steps. First,

$$\begin{aligned}
\int \phi^2 |X|^2 &= -2\int \phi R \langle \nabla\phi, X\rangle - \int \phi^2 R \triangle R \\
\int \phi^2 |X|^2 &\leq 4\int |\nabla\phi|^2 R^2 + 2\left(\int R^2\right)^{\frac{1}{2}} \left(\int \phi^4 |\triangle R|^2\right)^{\frac{1}{2}} \qquad (25)
\end{aligned}$$

Then we estimate the last term, using $|\triangle R|^2 = R_{,m\bar{m}} R_{,k\bar{k}}$. We get

$$\begin{aligned}
\int \phi^4 |\triangle R|^2 &= -4\int \phi^3 \triangle R \langle \nabla\phi, X\rangle - \int \phi^4 R_{,m\bar{m}k} R_{\bar{k}} \\
&= -4\int \phi^3 \triangle R \langle \nabla\phi, X\rangle + \int \phi^4 \mathrm{Ric}_{k\bar{s}} R_{,s} R_{\bar{k}} \\
\int \phi^4 |\triangle R|^2 &\leq 16\int \phi^2 |\nabla\phi|^2 |X|^2 + 2\int \phi^4 \,\mathrm{Ric}(X, X).
\end{aligned}$$

It is also possible to estimate the $\mathrm{Ric}(X, X)$ term:

$$\begin{aligned}
&\int \phi^4 \,\mathrm{Ric}(X, X) \\
&= -4\int \phi^3 R \,\mathrm{Ric}_{k\bar{s}} \phi_{,s} R_{,\bar{k}} - \int \phi^4 R \,\mathrm{Ric}_{k\bar{s},s} R_{,\bar{k}} - \int \phi^4 R \,\mathrm{Ric}_{k\bar{s}} R_{,\bar{k}s} \\
&\leq 2\int \phi^2 |\nabla\phi|^2 |X|^2 + 2\int \phi^4 |R|^2 |\mathrm{Ric}|^2 - \int \phi^4 R |X|^2 + \frac{1}{4}\int \phi^4 |\nabla X|^2
\end{aligned}$$

Finally we have to estimate the $|\nabla X|^2$ term.

$$\begin{aligned}\int \phi^4|\nabla X|^2 &= -4\int \phi^3 R_{,i\bar{j}}\phi_{,j}R_{,\bar{i}} - \int \phi^4 R_{,i\bar{j}j}R_{,\bar{i}} \\ &= -4\int \phi^3 R_{,i\bar{j}}\phi_{,j}R_{,\bar{i}} + \int \phi^4 \mathrm{Ric}_{i\bar{s}}R_{,s}R_{,\bar{i}} \\ \int \phi^4|\nabla X|^2 &\leq 16\int \phi^2|\nabla\phi|^2|X|^2 + 2\int \phi^4\,\mathrm{Ric}(X,X).\end{aligned}$$

Now we successively put these estimates back. First we get

$$\int \phi^4\,\mathrm{Ric}(X,X) \leq 12\int \phi^2|\nabla\phi|^2|X|^2 + 4\int \phi^4|R|^2|\,\mathrm{Ric}\,|^2 - 2\int \phi^4 R|X|^2.$$

Note that this also provides

$$\int \phi^4|\nabla X|^2 \leq 40\int \phi^2|\nabla\phi|^2|X|^2 + 8\int \phi^4|R|^2|\,\mathrm{Ric}\,|^2 - 4\int \phi^4 R|X|^2$$

and

$$\begin{aligned}&\int \phi^4|\triangle R|^2 \\ &\leq 40\int \phi^2|\nabla\phi|^2|X|^2 + 8\int \phi^4|R|^2|\,\mathrm{Ric}\,|^2 - 4\int \phi^4 R|X|^2. \qquad (26)\end{aligned}$$

Using the Sobolev inequality, we can do something with the final term:

$$\begin{aligned}\int \phi^4 R|X|^2 &\leq \left(\int R^{\frac{n}{2}}\right)^{\frac{2}{n}}\left(\int \phi^{4\gamma}|X|^{2\gamma}\right)^{\frac{1}{\gamma}} \\ &\leq 4C_S\left(\int R^{\frac{n}{2}}\right)^{\frac{2}{n}}\int \phi^2|\nabla\phi|^2|X|^2 + 2C_S\left(\int R^{\frac{n}{2}}\right)^{\frac{2}{n}}\int \phi^4|\nabla X|^2 \\ &\leq 4C_S\left(\int R^{\frac{n}{2}}\right)^{\frac{2}{n}}\int \phi^2|\nabla\phi|^2|X|^2 + 16C_S\left(\int R^{\frac{n}{2}}\right)^{\frac{2}{n}}\int \phi^4|R|^2|\,\mathrm{Ric}\,|^2 \\ &\quad -8C_S\left(\int R^{\frac{n}{2}}\right)^{\frac{2}{n}}\int \phi^4 R|X|^2 \\ \int \phi^4 R|X|^2 &\leq C\left(\int R^{\frac{n}{2}}\right)^{\frac{2}{n}}\int \phi^2|\nabla\phi|^2|X|^2 + C\left(\int R^{\frac{n}{2}}\right)^{\frac{2}{n}}\int \phi^4|R|^2|\,\mathrm{Ric}\,|^2\end{aligned}$$

Remarkably the constant C is bounded independently of the Sobolev constant. Thus

$$\int \phi^4|\triangle R|^2 \leq C\int \phi^2|\nabla\phi|^2|X|^2 + C\int \phi^4|R|^2|\,\mathrm{Ric}\,|^2. \qquad (27)$$

Returning to (25), we get

$$\begin{aligned}\int \phi^2|X|^2 &\leq 4\int |\nabla\phi|^2R^2 + C\left(\int R^2\right)^{\frac{1}{2}}\left(\int \phi^2|\nabla\phi|^2|X|^2\right)^{\frac{1}{2}} \\ &\quad +C\left(\int R^2\right)^{\frac{1}{2}}\left(\int \phi^4|R|^2|\,\mathrm{Ric}\,|^2\right)^{\frac{1}{2}}.\end{aligned}$$

Using $|\nabla \phi| \leq \frac{2}{r}$ gives us

$$\int \phi^2 |X|^2 \;\; \leq \;\; Cr^{-2} \int R^2 \, + \, C \left(\int R^2 \right)^{\frac{1}{2}} \left(\int \phi^4 |R|^2 |\operatorname{Ric}|^2 \right)^{\frac{1}{2}}. \qquad (28)$$

We must deal with the final term. In the case $n \geq 8$, we can easily deal with the final term:

$$\begin{aligned} \int \phi^4 |R|^2 |\operatorname{Ric}|^2 \;\; &\leq \;\; \left(\int |\operatorname{Ric}|^{\frac{n}{2}} \right)^{\frac{4}{n}} \left(\int |R|^{\frac{2n}{n-4}} \right)^{\frac{n-4}{n}} \\ &\leq \;\; (\operatorname{Vol} \operatorname{supp} \phi)^{\frac{n-8}{n}} \left(\int |\operatorname{Ric}|^{\frac{n}{2}} \right)^{\frac{4}{n}} \left(\int |R|^{\frac{n}{2}} \right)^{\frac{4}{n}} \end{aligned}$$

The case $n = 6$ is more difficult. Hölder gives

$$\int \phi^4 |\operatorname{Ric}|^2 |R|^2 \;\; \leq \;\; \left(\int \phi^4 |\operatorname{Ric}|^3 \right)^{\frac{2}{3}} \left(\int \phi^4 |R|^6 \right)^{\frac{1}{3}},$$

and we use the Sobolev inequality to get

$$\left(\int \phi^4 |R|^6 \right)^{\frac{2}{3}} \;\; \leq \;\; 4C_S \int \phi^2 |\nabla \phi|^2 |R|^4 \, + \, 4C_S \int \phi^4 |R|^2 |\nabla R|^2.$$

Now integration by parts on the last term yields

$$\left(\int \phi^4 |R|^6 \right)^{\frac{2}{3}} \;\; \leq \;\; 16C_S \int \phi^2 |\nabla \phi|^2 |R|^4 \, - \, 4C_S \int \phi^4 R^3 \triangle R.$$

Using Hólder's inequality and (27) gives

$$\begin{aligned} \left(\int \phi^4 |R|^6 \right)^{\frac{2}{3}} \;\; &\leq \;\; 16C_S \int |\nabla \phi|^2 |R|^4 \, + \, 4C_S \left(\int \phi^4 R^6 \right)^{\frac{1}{2}} \left(\int \phi^4 |\triangle R|^2 \right)^{\frac{1}{2}} \\ &\leq \;\; C \int |\nabla \phi|^2 |R|^4 \, + \, C \left(\int \phi^4 R^6 \right)^{\frac{1}{2}} \left(\int \phi^2 |\nabla \phi|^2 |X|^2 \, + \, \int \phi^4 |R|^2 |\operatorname{Ric}|^2 \right)^{\frac{1}{2}} \\ \left(\int \phi^4 |R|^6 \right)^{\frac{2}{3}} \;\; &\leq \;\; C \int |\nabla \phi|^4 |R|^2 \, + \, C \left(\int \phi^2 |\nabla \phi|^2 |X|^2 \right)^2 \, + \, C \left(\int \phi^4 |R|^2 |\operatorname{Ric}|^2 \right)^2 \end{aligned}$$

Putting this back in, we get

$$\int \phi^4 |\operatorname{Ric}|^2 |R|^2 \;\; \leq \;\; C \left(\int \phi^4 |\operatorname{Ric}|^3 \right)^{\frac{2}{3}} \int |\nabla \phi|^4 |R|^2 \, + \, C \left(\int \phi^4 |\operatorname{Ric}|^3 \right)^{\frac{2}{3}} \int \phi^2 |\nabla \phi|^2 |X|^2.$$

Finally we work with the case $n = 4$. We use simply

$$\int \phi^4 |\operatorname{Ric}|^2 |R|^2 \;\; \leq \;\; \left(\int \phi^4 |\operatorname{Ric}|^4 \right)^{\frac{1}{2}} \left(\int \phi^4 R^4 \right)^{\frac{1}{2}},$$

and use the Sobolev inequality to get

$$\left(\int \phi^4|\operatorname{Ric}|^4\right)^{\frac{1}{2}} \;\leq\; 2C_S\int |\nabla\phi|^2|\operatorname{Ric}|^2 + 2C_S\int \phi^2|\nabla\operatorname{Ric}|^2.$$

Using integration by parts on the last term lets us obtain

$$\left(\int \phi^4|\operatorname{Ric}|^4\right)^{\frac{1}{2}} \;\leq\; C\int |\nabla\phi|^2|\operatorname{Ric}|^2 + C\int \phi^2|\operatorname{Rm}||\operatorname{Ric}|^2 + C\int \phi^2|\operatorname{Ric}||\nabla X|.$$

Using our expressions for $\int \phi^4|\nabla X|^2$ and $\int R|X|^2$ allows us to obtain

$$\left(\int \phi^4|\operatorname{Ric}|^4\right)^{\frac{1}{2}} \;\leq\; C\int |\nabla\phi|^2|\operatorname{Ric}|^2 + C\left(\int |\operatorname{Ric}|^2\right)^{\frac{1}{2}}\left(\int \phi^2|\nabla\phi|^2|X|^2\right)^{\frac{1}{2}}.$$

The Sobolev inequality applied to $\int \phi^4|R|^4$ gives us

$$\left(\int \phi^4R^4\right)^{\frac{1}{2}} \;\leq\; C\int |\nabla\phi|^2R^2 + C\int \phi^2|X|^2$$

Putting the estimates for $\int \phi^4R^2|\operatorname{Ric}|^2$ in the cases $n=4$, $n=6$, and $n\geq 8$ into (28) lets us conclude, regardless of dimension, that

$$\int \phi^2|X|^2 \;\leq\; Cr^{-2}\int R^2 + r^{n-6}\left(\int R^{\frac{n}{2}}\right)^{\frac{4}{n}}$$

The conclusion now follows from Proposition 3.5. □

Proposition 4.3 *If $p>\frac{n}{2}$ and $B(o,r)$ consists of smooth points, there exists $\epsilon_0=\epsilon_0(p,C_S,n)$ and $C=C(p,C_S,n)$ so that $\int_{B(o,r)}|\operatorname{Rm}|^{\frac{n}{2}}\leq\epsilon_0$ implies*

$$\left(\int_{B(o,r/2)}|\operatorname{Ric}|^p\right)^{\frac{1}{p}} \;\leq\; Cr^{\frac{n}{p}-2}\left(\int_{B(o,r/2)}|\operatorname{Ric}|^{\frac{n}{2}}\right)^{\frac{2}{n}}$$

Pf

We use integration by parts to get

$$\begin{aligned}\int \phi^l|\nabla X|^k &= \int \phi^l|\nabla X|^{k-2}\langle\nabla X,\nabla X\rangle\\ &= -l\int \phi^{l-1}|\nabla X|^{k-2}\langle\nabla\phi\otimes X,\nabla X\rangle\\ &\quad -(k-2)\int \phi^l|\nabla X|^{k-3}\langle\nabla|\nabla X|\otimes X,\nabla X\rangle\\ &\quad -\phi^l\int |\nabla X|^{k-2}\langle X,\triangle X\rangle\end{aligned}$$

Using Hölder's inequality on the first term and using $\nabla^2 X = \mathrm{Rm} * X$, we get

$$\int \phi^l |\nabla X|^k \quad \leq \quad C(k,l) \int \phi^{l-2} |\nabla \phi|^2 |\nabla X|^{k-2} |X|^2 \,+\, C(k,l) \phi^l \int |\nabla X|^{k-2} |X|^2 |\,\mathrm{Rm}\,|$$

Assuming $k < n$ we can use Hölder's inequality again to get

$$\int \phi^l |\nabla X|^k \quad \leq \quad \left(\int \phi^{\frac{ln}{n-k}} |X|^{\frac{kn}{n-k}} \right)^{\frac{n-k}{n}}. \tag{29}$$

This holds in particular when $k = \frac{n}{2}$. Now the inequality $\triangle|\,\mathrm{Ric}\,| \geq -C|\,\mathrm{Rm}\,||\,\mathrm{Ric}\,| - C|\nabla X|$ yields the conclusion, via Proposition 3.5. $\square$

Proposition 4.4 *If $p > \frac{n}{2}$ and $B(o,r)$ consists of smooth points, there exists $\epsilon_0 = \epsilon_0(p, C_S, n)$ and $C = C(p, C_S, n)$ so that $\int_{B(o,r)} |\,\mathrm{Rm}\,|^{\frac{n}{2}} \leq \epsilon_0$ implies*

$$\left(\int_{B(o,r/2)} |\,\mathrm{Rm}\,|^p \right)^{\frac{1}{p}} \quad \leq \quad C r^{\frac{n}{p}-2} \left(\int_{B(o,r/2)} |\,\mathrm{Rm}\,|^{\frac{n}{2}} \right)^{\frac{2}{n}}$$

<u>*Pf*</u>

Following the calculation leading up to (47), we get that

$$\begin{aligned} &\left(\int \phi^{k\gamma} |\,\mathrm{Rm}\,|^{k\gamma} \right)^{\frac{1}{\gamma}} \\ &\leq C \int \phi^{l-2} |\nabla \phi|^2 |\,\mathrm{Rm}\,|^k \,+\, C \int \phi^l |\,\mathrm{Rm}\,|^{k+1} \,+\, \int \phi^l |\,\mathrm{Rm}\,|^{k-1} |\nabla X| \end{aligned}$$

holds when $\mathrm{supp}\,\phi$ consists of smooth points. The second term on the right easily combines into the left side when $\int_{\mathrm{supp}\,\phi} |\,\mathrm{Rm}\,|^{\frac{n}{2}}$ is small, and then using Hölder's inequality on the rightmost term, we get

$$\left(\int \phi^{k\gamma} |\,\mathrm{Rm}\,|^{k\gamma} \right)^{\frac{1}{\gamma}} \quad \leq \quad C \int \phi^{l-2} |\nabla \phi|^2 |\,\mathrm{Rm}\,|^k \,+\, \left(\int \phi^{\square} |\nabla X|^{\frac{nk}{2k+n-2}} \right)^{\frac{2k+n-2}{n}}.$$

Noticing that $\frac{nk}{2k+n-2} < n$ and using (29) gives

$$\left(\int \phi^{k\gamma} |\,\mathrm{Rm}\,|^{k\gamma} \right)^{\frac{1}{\gamma}} \quad \leq \quad C r^{-2} \int \phi^{l-2} |\,\mathrm{Rm}\,|^k.$$

Iterating this inequality yields the conclusion. $\square$

Proposition 4.5 *If $p > \frac{n}{2}$ and $B(o,r)$ consists of smooth point, there exists $\epsilon_0 = \epsilon_0(p, C_S, n)$ and $C = C(p, C_S, n)$ so that $\int_{B(o,r)} |\operatorname{Rm}|^{\frac{n}{2}} \leq \epsilon_0$ implies*

$$\begin{aligned}\left(\int_{B(o,r/2)} |\nabla \operatorname{Ric}|^p\right)^{\frac{1}{p}} &\leq Cr^{\frac{n}{k}-3}\left(\int_{B(o,r/2)} |\operatorname{Ric}|^{\frac{n}{2}}\right)^{\frac{2}{n}} \\ \left(\int_{B(o,r/2)} |\nabla X|^p\right)^{\frac{1}{p}} &\leq Cr^{\frac{n}{k}-4}\left(\int_{B(o,r/2)} |R|^{\frac{n}{2}}\right)^{\frac{2}{n}}\end{aligned}$$

_Pf_

Applying the Sobolev inequality, integration by parts, and the elliptic inequality for $|\operatorname{Rm}|$, we get

$$\begin{aligned}C\left(\int \phi^{l\gamma}|\nabla \operatorname{Ric}|^{k\gamma}\right)^{\frac{1}{\gamma}} &\leq \int \phi^{l-2}|\nabla\phi|^2|\nabla \operatorname{Ric}|^k \\ &+ \int \phi^l|\nabla \operatorname{Ric}|^{k-2}|\operatorname{Ric}|^2|\operatorname{Rm}|^2 + \int \phi^l|\nabla \operatorname{Ric}|^{k-2}|\nabla X|^2.\end{aligned}$$

Hölder's inequality, combined with Proposition 4.4 gives

$$\left(\int \phi^{l\gamma}|\nabla \operatorname{Ric}|^{k\gamma}\right)^{\frac{1}{\gamma}} \leq Cr^{-2}\int \phi^{l-2}|\nabla \operatorname{Ric}|^k + \left(\int \phi^{\Box}|\nabla X|^{\frac{nk}{k+n-2}}\right)^{\frac{n}{k+n-2}}$$

Since $\frac{nk}{k+n-2} < n$, using (29), we get

$$\left(\int \phi^{l\gamma}|\nabla \operatorname{Ric}|^{k\gamma}\right)^{\frac{1}{\gamma}} \leq Cr^{-2}\int \phi^{l-2}|\nabla \operatorname{Ric}|^k,$$

which we can iterate to get the stated result for $|\nabla \operatorname{Ric}|$. Now the equation $\triangle|\nabla X| \geq -C|\nabla \operatorname{Ric}||X| - C|\operatorname{Rm}||\nabla X|$, along with Proposition 3.5 (which always works at smooth points), yields the result for $|\nabla X|$. □

4.2 Pointwise curvature regularity

Here we assume that Proposition 4.1 has been entirely proved at smooth points. The beginning of the proof was undertaken in the previous section. The rest of the proof, consisting of an induction argument in dimension, is in the appendix.

Theorem 4.6 *Assume $B(o,r)$ consists of manifold points. There exists an $\epsilon_0 = \epsilon_0(C_S, n, p)$ and $C = C(C_S, n, p)$ so that $\int_{B_r} |\operatorname{Rm}|^{\frac{n}{2}} \leq \epsilon_0$ implies*

$$\sup_{B(o,r/2)} |\nabla^p \operatorname{Rm}| \leq Cr^{-p-2}\left(\int_{B(o,r)} |\operatorname{Rm}|^{\frac{n}{2}}\right)^{\frac{2}{n}}.$$

Pf

First we prove a commutator formula. If T is any tensor, we have

$$\begin{aligned} \triangle \nabla T &= T_{,imm} = T_{,mim} + (\mathrm{Rm}_{im**} * T)_{,m} \\ &= T_{,mmi} + \mathrm{Rm} * \nabla T + \mathrm{Rm}_{im**,m} * T + \mathrm{Rm}_{im**} * T_{,m} \\ &= \nabla \triangle T + \mathrm{Rm} * \nabla T + \nabla \mathrm{Ric} * T. \end{aligned} \tag{30}$$

Here the stars in the subscript positions of Rm are meant to indicate a contraction with various indices of T. Replacing T with $\nabla^{p-1}T$, an induction argument gives

$$[\triangle, \nabla^p] = \sum_{i=0}^{p-1} \nabla^i \mathrm{Rm} * \nabla^{p-i} + \sum_{i=1}^{q} \nabla \mathrm{Ric} * \nabla^{p-i}.$$

Therefore

$$\triangle \nabla^p \mathrm{Rm} = \sum_{i=0}^{p} \nabla^i \mathrm{Rm} * \nabla^{p-i} \mathrm{Rm} + \nabla^{p+2} \mathrm{Ric},$$

so

$$\begin{aligned} \triangle |\nabla^p \mathrm{Rm}| \geq & -C|\mathrm{Rm}||\nabla^p \mathrm{Rm}| \\ & - C \sum_{i=1}^{p-1} |\nabla^i \mathrm{Rm}||\nabla^{p-i} \mathrm{Rm}| - C|\nabla^{p+2} \mathrm{Ric}|. \end{aligned}$$

With $u = |\nabla^p \mathrm{Rm}|$, $f = C|\mathrm{Rm}|$, and $g = C\sum_{i=1}^{p-1} |\nabla^i \mathrm{Rm}||\nabla^{p-i} \mathrm{Rm}| + C|\nabla^{p+2} \mathrm{Ric}|$, we get the elliptic inequality

$$\triangle u \geq -f\,u - g$$

which holds everywhere that $u \neq 0$. Proposition (4.1) gives that $f, g \in L^{s'}(B(o, r/2))$ for some $s' > n/2$, so theorem 8.15 of [GT] gives

$$\sup_{B(o,r/4)} |\nabla^p \mathrm{Rm}| \leq Cr^{-2} \left(\int_{B(o,r/2)} |\nabla^p \mathrm{Rm}|^{\frac{n}{2}} \right)^{\frac{2}{n}},$$

and so

$$\sup_{B(o,r/4)} |\nabla^p \mathrm{Rm}| \leq Cr^{-p-2} \left(\int_{B(o,r)} |\mathrm{Rm}|^{\frac{n}{2}} \right)^{\frac{2}{n}}.$$

Applying this for balls $B(o', r/2)$ with $o' \in \partial B(o, 3r/8)$, we get the final form of the result. □

4.3 Removing curvature singularities, $n \geq 6$

Here we undertake the proof Proposition 4.1 in the cases where $B(o, r)$ has curvature singularities and the dimension satisfies $n \geq 6$. We will make use of our original elliptic system

$$\begin{aligned} \triangle \operatorname{Rm} &= \operatorname{Rm} * \operatorname{Rm} + \nabla^2 \operatorname{Ric} && (31) \\ \triangle \operatorname{Ric} &= \operatorname{Rm} * \operatorname{Ric} + \nabla X && (32) \\ \triangle X &= \operatorname{Ric} * X. && (33) \end{aligned}$$

In addition we will use the formulas

$$\begin{aligned} \nabla^2 X &= \operatorname{Rm} * X && (34) \\ \triangle \nabla X &= \nabla \operatorname{Ric} * X \, + \, \operatorname{Rm} * \nabla X && (35) \\ \triangle \nabla \operatorname{Ric} &= \operatorname{Rm} * \nabla \operatorname{Ric} \, + \, \operatorname{Ric} * \nabla \operatorname{Rm} \, + \, \operatorname{Rm} * X. && (36) \end{aligned}$$

Our ultimate goal is to show that $|\operatorname{Rm}| \in L^k$ for all k despite the singularities. One must only show that $|\nabla^2 \operatorname{Ric}| \in L^{\frac{n}{2}}$, and then Proposition 3.5 theory gives $|\operatorname{Rm}| \in L^k$. Showing that $|\nabla^2 \operatorname{Ric}| \in L^{\frac{n}{2}}$ isn't too bad at smooth points, but with singularities we must use a more round-about route. We already have $|X| \in L^k$ (Proposition 3.5). We can show $|\nabla X| \in L^n$, so Proposition 3.5 gives that $|\operatorname{Ric}| \in L^k$.

Now at this stage we try to get estimates for Rm. The model case is the real-valued system in divergence form

$$\triangle u \quad \geq \quad -fu \, - \, \nabla^i g_i,$$

where one gets that $u \in L^k$ provided $f \in L^{\frac{n}{2}}$ and $g_i \in L^n$. Abusing both notation and the very notion of divergence, we consider equation (31) to have a nonhomogeneous term in divergence form, namely $g_i = \nabla \operatorname{Ric}$. If $|\nabla \operatorname{Ric}| \in L^n$ we then expect $|\operatorname{Rm}| \in L^k$. This intuition certainly pans out in the smooth case, but unfortunately the tool in the singular case, Proposition 3.5, is not built to handle the divergence term. It is essentially the divergence structure that we exploit in our argument, however, so it is likely that some improvements can be made to Proposition 3.5 also.

Technically intricate arguments allow us to play estimates for $|\nabla \operatorname{Ric}|$ and $|\operatorname{Rm}|$ off of each other; we show that $|\operatorname{Rm}| \in L^p$ implies an improved estimate for $|\nabla \operatorname{Ric}|$, and this improved estimate in turn lets us bootstrap $|\operatorname{Rm}|$ into higher L^p spaces.

In we use the following the shorthand notation: if p is a number we use p^- to indicate a variable that may have any value less than p, and p^+ to indicate a variable that may have any value greater than p.

Lemma 4.7 *Assume M is a manifold-with-singularities. There exist $\epsilon_0 = \epsilon_0(n,k,C_S)$ and $C = C(n,k,C_S)$ so that $\int_{B(o,r)} |\operatorname{Ric}|^{\frac{n}{2}} \leq \epsilon_0$ implies*

$$\left(\int_{B(o,r/2)} |X|^k\right)^{\frac{1}{k}} \quad \leq \quad C r^{\frac{n}{k}-3} \left(\int_{B(o,r)} |R|^{\frac{n}{2}}\right)^{\frac{2}{n}}.$$

<u>*Pf*</u>

The proof of this lemma in the smooth case, given by Proposition 4.2, will carry through provided we can justify the use of integration by parts in case $n \geq 6$.

Assume that $|X| \in L^{p^-}_{loc}$. Assuming that $\operatorname{supp}\phi$ consists of smooth points, we get

$$\begin{aligned}
\int \phi^p |X|^p \quad &= \quad -p \int \phi^{p-1} R \left\langle \nabla\phi, X \right\rangle |X|^{p-2} \\
&\quad - (p-2) \int \phi^p R \left\langle \nabla|X|, X \right\rangle |X|^{p-3} \; - \; \int \phi^p R \triangle R |X|^{p-2} \\
&\leq \quad p \int \phi^{p-1} |R| |\nabla\phi| |X|^{p-1} \\
&\quad + (p-2) \int \phi^p |R| |X|^{p-2} |\nabla X| \; + \; \int \phi^p |R| |\triangle R| |X|^{p-2} \\
&\leq \quad C|R| \left(\int |\nabla\phi|^n\right)^{\frac{1}{n}} \left(\int |X|^{(p-1)\frac{n}{n-1}}\right)^{\frac{n-1}{n}} \\
&\quad + C|R| \left(\int |\nabla X|^2\right)^{\frac{1}{2}} \left(\int |X|^{2p-4}\right)^{\frac{1}{2}}
\end{aligned}$$

Now replace ϕ by $\phi \cdot \phi_R$ where $\phi \equiv 1$ across the singularity o, and ϕ_R is a cutoff function with $\phi_R \equiv 0$ inside $B(o,R)$, $\phi_R \equiv 1$ outside $B(o,2R)$, and $|\nabla\phi_R| \leq \frac{2}{R}$. Then we can take a limit as $R \to 0$ assuming first that $(p-1)\frac{n}{n-1} < p$ and second that $2p-4 < p$; it suffices to require $p < 4$.

Therefore, assuming $p < 4$ so that integration by parts works, we can get

$$\int \phi^p |X|^p \quad \leq \quad C \int |\nabla\phi|^p |R|^p \; + \; C \int \phi^p |R|^{\frac{p}{2}} |\nabla X|^{\frac{p}{2}}.$$

Now we can take a limit as $p \to 4$. Using the Dominated Convergence Theorem on the right side and Fatou's lemma on the left, we get that this inequality holds for $p = 4$ as well. Now we can repeat the proof of Proposition 4.2. □

Lemma 4.8 *Assume M is a manifold-with-singularities. If $2 \leq k \leq n$ there* *exist* $\epsilon_0 = \epsilon_0(n,k,C_S)$ *and* $C = C(n,k,C_S)$ *so that* $\int_{B(o,r)} |\operatorname{Rm}|^{\frac{n}{2}} \le \epsilon_0$ *implies*

$$\left(\int_{B(o,r/2)} |\nabla X|^k\right)^{\frac{1}{k}} \quad \le \quad C r^{\frac{n}{k}-4} \left(\int_{B(o,r)} |R|^{\frac{n}{2}}\right)^{\frac{2}{n}},$$

irrespective of the presence of singularities.

<u>*Pf*</u>

$$\begin{aligned}\int \phi^l |\nabla X|^k \quad &= \quad -l \int \phi^{l-1} |\nabla X|^{k-2} \langle \nabla X, \nabla\phi \otimes X\rangle \\ &\quad -(k-2)\int \phi^l |\nabla X|^{k-3} \langle \nabla X, \nabla|\nabla X| \otimes X\rangle \\ &\quad - \int \phi^l |\nabla X|^{k-2} \langle \triangle X, X\rangle\end{aligned}$$

We use $\nabla^2 X \;=\; \operatorname{Rm} * X$ and $\triangle X \;=\; \operatorname{Ric} * X$ to get

$$\begin{aligned}\int \phi^l |\nabla X|^k \quad &\le \quad \frac{l^2}{2}\int \phi^{l-2}|\nabla\phi|^2|\nabla X|^{k-2}|X|^2 + \frac{1}{2}\int \phi^l |\nabla X|^k \\ &\quad +(k-2)\int \phi^l|\nabla X|^{k-2}|X|^2|\operatorname{Rm}| \\ &\quad + \int \phi^l |\nabla X|^{k-2}|X|^2|\operatorname{Ric}|,\end{aligned}$$

so with $C = C(k,l)$ we have

$$\begin{aligned}\int \phi^l|\nabla X|^k \quad &\le \quad C\int \phi^{l-2}|\nabla\phi|^2|\nabla X|^{k-2}|X|^2 \\ &\quad + C\int \phi^l |\nabla X|^{k-2}|X|^2|\operatorname{Rm}|. \end{aligned} \tag{37}$$

If $\operatorname{supp}\phi$ has a singularity, we now show that when $k < 2p \le n$ this still holds. We get

$$\begin{aligned}\int \phi^l|\nabla X|^k \quad &\le \quad C\left(\int \phi^l|\nabla X|^k\right)^{\frac{k-2}{k}} \left(\int \phi^{l-k}|\nabla\phi|^k|X|^k\right)^{\frac{2}{k}} \\ &\quad + C\left(\int \phi^l|\nabla X|^k\right)^{\frac{k-2}{k}} \left(\int \phi^{\frac{2pl}{2p-k}}|X|^{\frac{2pk}{2p-k}}\right)^{\frac{2p-k}{pk}} \left(\int |\operatorname{Rm}|^p\right)^{\frac{1}{p}}\end{aligned} \tag{38}$$

Replace ϕ in the inequality by $\phi\cdot\phi_R$. Assuming o is a singularity we choose the cutoff function ϕ_R with the following properties: $\phi_R \equiv 1$ outside $B(o,2R)$, $\phi \equiv 0$ inside $B(o,R)$ and $|\nabla\phi_R| \le \frac{2}{R}$. If we take a limit as $R \to 0$ (so the cutoff function closes in around the singularity), we get that $\phi\cdot\phi_R \to \phi$ and we can use the dominated convergence theorem on everything except the integral with the $\nabla(\phi\phi_R)$, which we analyze separately. With $k < n$ we get

$$\begin{aligned}\int (\phi\cdot\phi_R)^{l-k}|\nabla(\phi\cdot\phi_R)|^k|X|^k \quad &\leq \quad \left(\int|\nabla\phi_R|^n\right)^{\frac{k}{n}}\left(\int_{\mathrm{supp}\,\nabla\phi_1}\phi^{\frac{ln}{n-k}}|X|^{\frac{kn}{n-k}}\right)^{\frac{n-k}{n}} \\ &\quad + \int\phi_R^k|\nabla\phi|^k|X|^k.\end{aligned}$$

Since $\int|\nabla\phi_R|^n$ is bounded and $|X| \in L^{\frac{nk}{n-k}}$ by Lemma 4.7, the first term on the right side goes to zero as $R \to 0$. Therefore (38) holds despite the possible presence of singularities. Using $p = \frac{n}{2}$ in (38) now gives

$$\int\phi^l|\nabla X|^k \quad\leq\quad C\int\phi^{l-k}|\nabla\phi|^k|X|^k + C\left(\int\phi^{\frac{nl}{n-k}}|X|^{\frac{nk}{n-k}}\right)^{\frac{n-k}{n}},$$

where $C = C(k,l)$, and Proposition 3.5 yields finally (with $k < n$)

$$\int_{B(o,r/2)}|\nabla X|^k \quad\leq\quad Cr^{-k}\int_{B(o,r)}|X|^k.$$

The value of C does not degenerate as $k \nearrow n$, so we can take a limit, using Fatou's lemma on the left side, and get the result for $k = n$ as well. $\square$

Theorem 4.9 *Assume M is a manifold-with-singularities. For $\frac{n}{n-2} \leq k \leq a < \infty$, there exist $\epsilon_0 = \epsilon_0(C_S, n, a, k)$ and $C = C(C_S, n, a, k)$ so that $\int_{B(o,r)}|\,\mathrm{Rm}\,|^{\frac{n}{2}} \leq \epsilon_0$ implies*

$$\left(\int_{B(o,r/2)}|\,\mathrm{Ric}\,|^a\right)^{\frac{1}{a}} \quad\leq\quad Cr^{\frac{n}{a}-\frac{n}{k}}\left(\int_{B(o,r)}|\,\mathrm{Ric}\,|^k\right)^{\frac{1}{k}},$$

irrespective of the presence of singularities.

<u>*Pf*</u>
With

$$\begin{aligned}\triangle\,\mathrm{Ric} \quad &= \quad \mathrm{Rm} * \mathrm{Ric} + \nabla X && (39)\\ \triangle|\,\mathrm{Ric}\,| \quad &\geq \quad -|\,\mathrm{Rm}\,||\,\mathrm{Ric}\,| - |\nabla X|, && (40)\end{aligned}$$

and since $|\nabla X| \in L^{\frac{n}{2}}$, we can use Proposition 3.5 to get that $|\,\mathrm{Ric}\,| \in L^k$ for all $k < \infty$. We get the local estimates

$$\left(\int_{B(o,r/2)}|\,\mathrm{Ric}\,|^a\right)^{\frac{1}{a}} \quad\leq\quad Cr^{\frac{n}{a}-\frac{n}{k}}\left(\int_{B(o,r)}|\,\mathrm{Ric}\,|^k\right)^{\frac{1}{k}}. \qquad (41)$$

□

Note that the hypotheses of the following technical lemma hold because we have independently proven Theorem 6.5 and Theorem 4.6 in the smooth case. The proof is just a more involved version of the proof of 4.7.

Technical Lemma 4.10 *Assuming* $|\nabla^p \operatorname{Ric}| = o(r^{-2-p})$ *near singularities, then* $|\nabla^2 \operatorname{Ric}| \in L^{\frac{n}{3}^-}$ *and* $|\nabla \operatorname{Ric}| \in L^{\frac{2}{3}n^-}$.

<u>*Pf*</u>

First, we know that $|\nabla^4 \operatorname{Ric}| \in L^{\frac{n}{6}^-}$. Thus assuming $|\nabla^2 \operatorname{Ric}| \in L^{p^-}$ we have

$$\begin{aligned}
\int \phi^2 |\nabla^3 \operatorname{Ric}|^k &= -2\int \phi |\nabla^3 \operatorname{Ric}|^{k-2} \left\langle \nabla\phi \otimes \nabla^2 \operatorname{Ric}, \nabla^3 \operatorname{Ric}\right\rangle \\
&\quad -(k-2)\int \phi^2 |\nabla^3 \operatorname{Ric}|^{k-3} \left\langle \nabla|\nabla^3 \operatorname{Ric}| \otimes \nabla^2 \operatorname{Ric}, \nabla^3 \operatorname{Ric}\right\rangle \\
&\quad -\int \phi^2 |\nabla^3 \operatorname{Ric}|^{k-3} \left\langle \nabla^2 \operatorname{Ric}, \triangle \nabla^2 \operatorname{Ric}\right\rangle \\
&\leq c\int \phi |\nabla\phi| |\nabla^3 \operatorname{Rm}|^{k-1} |\nabla^2 \operatorname{Ric}| \\
&\quad + c\left(\int |\nabla^3 \operatorname{Ric}|^{k^-}\right)^{\frac{k-2}{k}\frac{n}{n-6}^+} \left(\int |\nabla^2 \operatorname{Ric}|^{\frac{kn}{2n-6k}^+}\right)^{\frac{2n-6k}{k(n-6)}^-} \\
&\quad + c\int |\nabla^4 \operatorname{Ric}|^{\frac{n}{6}^-}
\end{aligned}$$

which holds across singularities provided $\frac{nk}{n+k} \leq p$. Thus $\frac{nk}{2n-6k} < p$ gives $|\nabla^3 \operatorname{Ric}| \in L^k$. Since we can always choose $p > \frac{n}{5}$, we have that

$$k < \frac{2np}{n+6p} \quad \Longrightarrow \quad |\nabla^3 \operatorname{Ric}| \in L^k.$$

We do the same thing for $|\nabla^2 \operatorname{Ric}|$. Assume $|\nabla \operatorname{Ric}| \in L^{q^-}$. We get

$$\begin{aligned}
\int \phi^2 |\nabla^2 \operatorname{Ric}|^k &\leq 2\int \phi |\nabla\phi| |\nabla^2 \operatorname{Ric}|^{k-1} |\nabla \operatorname{Ric}| \\
&\quad + c\left(\int |\nabla^2 \operatorname{Ric}|^{k^-}\right)^{\frac{k-2}{k}\frac{m}{m-1}^+} \left(\int |\nabla \operatorname{Ric}|^{\frac{km}{2m-k}^+}\right)^{\frac{2m-k}{k(m-1)}^-} \\
&\quad + (k-1)\int \phi^m |\nabla^3 \operatorname{Ric}|^{m^-}
\end{aligned}$$

This holds across singularities if $\frac{nk}{n+k} < q$. If $|\nabla^3 \operatorname{Ric}| \in L^{m^-}$ and $\frac{km}{2m-k} < q$, then $|\nabla^2 \operatorname{Ric}| \in L^k$. Assuming $m < n$, we get $k < \frac{2mq}{m+q}$ implies $|\nabla^2 \operatorname{Ric}| \in L^k$.

We do the same thing for $|\nabla \operatorname{Ric}|$. Assume $|\operatorname{Ric}| \in L^{q^-}$. We get

$$\begin{aligned}
\int \phi^2 |\nabla \operatorname{Ric}|^k \quad &\leq \quad 2\int \phi |\nabla \phi| |\nabla \operatorname{Ric}|^{k-1} |\operatorname{Ric}| \\
&\quad + c\left(\int |\nabla \operatorname{Ric}|^{k^-}\right)^{\frac{k-2}{k}\frac{m}{m-1}^+} \left(\int |\operatorname{Ric}|^{\frac{km}{2m-k}^+}\right)^{\frac{2m-k}{k(m-1)}^-} \\
&\quad + (k-1)\int \phi^m |\nabla^2 \operatorname{Ric}|^{m^-}
\end{aligned}$$

This holds across singularities if $\frac{nk}{n+k} < q$. If $|\nabla^2 \operatorname{Ric}| \in L^{m^-}$ and $\frac{km}{2m-k} < q$, then $|\nabla \operatorname{Ric}| \in L^k$. Assuming $m < n$, we get $k < \frac{2mq}{m+q}$ implies $|\nabla \operatorname{Ric}| \in L^k$.

The result of these three inequalities is that

$$|\nabla^3 \operatorname{Ric}| \in L^{m^-}, \text{ and } \quad |\nabla \operatorname{Ric}| \in L^{q^-} \implies |\nabla^2 \operatorname{Ric}| \in L^{\frac{2mq}{m+q}^-} \tag{42}$$

$$|\nabla^2 \operatorname{Ric}| \in L^{r^-} \implies |\nabla \operatorname{Ric}| \in L^{2r^-} \tag{43}$$

$$|\nabla^2 \operatorname{Ric}| \in L^{r^-} \implies |\nabla^3 \operatorname{Ric}| \in L^{\frac{2nr}{n+6r}} \tag{44}$$

Fixing m and iterating, we continue to get increases in q up until $q = 3m$. Thus we get $|\nabla \operatorname{Ric}| \in L^{3m^-}$ and $|\nabla^2 \operatorname{Ric}| \in L^{\frac{3m}{2}^-}$. Then letting m vary and iterating, we get improvements up until $m = \frac{2n}{9}$.

Therefore $|\nabla \operatorname{Ric}| \in L^{\frac{2}{3}n^-}$, $|\nabla^2 \operatorname{Ric}| \in L^{\frac{1}{3}n^-}$, $|\nabla^3 \operatorname{Ric}| \in L^{\frac{2}{9}n^-}$.

□

The next lemma sets up the possibility of using integration by parts across singularities, but does not give any particular bound for $L^p(|\operatorname{Rm}|)$.

Technical Lemma 4.11 *Assuming the above lemma, we have* $|\operatorname{Rm}| \in L^p$ *and* $|\nabla \operatorname{Ric}| \in L^p$ *for all* p.

<u>*Pf*</u>
We can use the improvement integral bounds of $|\nabla^2 \operatorname{Ric}|$ to our advantage. Sobolev's inequality gives

$$C\left(\int \phi^{2k} |\operatorname{Rm}|^{k\gamma}\right)^{\frac{1}{\gamma}} \quad \leq \quad \int |\nabla \phi|^2 |\operatorname{Rm}|^k + \int \phi^2 |\operatorname{Rm}|^{k-2} |\nabla |\operatorname{Rm}||^2.$$

Choosing $k = \frac{n-2}{2}$ so integration by parts works across singularities (by Lemma 3.3), so we get

$$C\left(\int \phi^{2k} |\operatorname{Rm}|^{k\gamma}\right)^{\frac{1}{\gamma}} \quad \leq \quad \int |\nabla \phi|^2 |\operatorname{Rm}|^k + \int \phi^2 |\operatorname{Rm}|^{k+1} + \int \phi^2 |\operatorname{Rm}|^{k-1} |\nabla^2 \operatorname{Ric}|.$$

Using that $\int |\operatorname{Rm}|^{\frac{n}{2}}$ is small we get

$$C\left(\int \phi^{2k}|\operatorname{Rm}|^{\frac{n}{2}}\right)^{\frac{1}{\gamma}} \quad \leq \quad \int |\nabla\phi|^2|\operatorname{Rm}|^{\frac{n-2}{2}} + C\left(\int |\nabla^2\operatorname{Ric}|^{\frac{n}{4}}\right)^{\frac{4}{n}}\left(\int |\operatorname{Rm}|^{\frac{n}{2}}\right)^{\frac{n-4}{n}}$$

Since $|\nabla^2\operatorname{Ric}| \in L^{\frac{n}{4}}$ we get

$$C\left(\int \phi^{2k}|\operatorname{Rm}|^{\frac{n}{2}}\right)^{\frac{1}{\gamma}} \quad \leq \quad \int |\nabla\phi|^2|\operatorname{Rm}|^{\frac{n-2}{2}} + \left(\int |\nabla^2\operatorname{Ric}|^{\frac{n}{4}}\right)^{\frac{2(n-2)}{n}}. \tag{45}$$

Since $\operatorname{Rm} \in L^{n/2}$ this holds across singularities. Using an argument similar to Theorem 5.8 of [BKN] we can get that $\int_{B_r} |\operatorname{Rm}|^{\frac{n}{2}}$ decays like $\left(\int_{B_r} |\nabla^2\operatorname{Ric}|^{\frac{n}{4}}\right)^2$ (the argument needed here is given in detail in Lemma 4.17). Since $|\nabla^2\operatorname{Ric}| \in L^{\frac{n}{3}^-}$, we get that

$$\int_{B_r} |\operatorname{Rm}|^{\frac{n}{2}} \quad = \quad O(r^{\frac{n}{2}^-}).$$

Using this in conjunction with Theorem 4.6 gives that $|\nabla^s\operatorname{Rm}| = O(r^{(-s-1)^-})$ near singularities. Note that all this fails in the case $n < 6$, for in that case the use of the Sobolev inequality that began the discussion would be unavailable to us. This means that $|\operatorname{Rm}| \in L^{n^-}$, $|\nabla\operatorname{Rm}| \in L^{\frac{n}{2}^-}$, $|\nabla^2\operatorname{Rm}| \in L^{\frac{n}{3}^-}$, $|\nabla^3\operatorname{Rm}| \in L^{\frac{n}{4}^-}$ etc.

Now we return to (42), (43), and (44) from above. We now have $|\nabla^3\operatorname{Ric}| \in L^{\frac{n}{4}}$ so we can expect some improvements. We initially have that $|\nabla\operatorname{Ric}| \in L^{\frac{2}{3}n^-}$, as we got from the last theorem. We can iterate up until $q = \frac{3}{4}n$, so that $|\nabla^2\operatorname{Ric}| \in L^{\frac{3n}{8}^-}$. We get therefore that

$$\left(\int_{B_r} |\nabla^2\operatorname{Ric}|^{\frac{n}{4}}\right)^2 = O(r^{\frac{2n}{3}^-})$$

and so (45) implies that $\int_{B(o,r)} |\operatorname{Rm}|^{\frac{n}{2}}$ decays like $O(r^{\frac{2n}{3}^-})$. Running through the above argument again, that $|\nabla^s\operatorname{Rm}| = O(r^{-(s+\frac{2}{3})^-})$. This yields actually that $|\nabla\operatorname{Rm}| \in L^{\frac{3}{5}n^-}$, and with

$$\triangle\nabla\operatorname{Ric} \quad = \quad \nabla\operatorname{Rm} * \operatorname{Ric} + \operatorname{Rm} * \nabla\operatorname{Ric},$$

Now Proposition 3.5 implies that $|\nabla\operatorname{Ric}| \in L^p$ for all p.

Then (42) implies now that $|\nabla^2\operatorname{Ric}| \in L^{\frac{6}{11}n^-}$. But then Proposition 3.5 applied to $\triangle\operatorname{Rm} = \operatorname{Rm} * \operatorname{Rm} + \nabla^2\operatorname{Ric}$ implies that $|\operatorname{Rm}| \in L^p$ for all p. □

Proposition 4.12 *Assuming* $\int_{B(o,r)} |\operatorname{Rm}|^{\frac{n}{2}} < \epsilon_0$ *we get*

$$\begin{aligned}\left(\int_{B(o,r/2)} |\operatorname{Rm}|^a\right)^{\frac{1}{a}} &\leq Cr^{\frac{n}{a}-2}\left(\int_{B(o,r)} |\operatorname{Rm}|^{\frac{n}{2}}\right)^{\frac{2}{n}}\\ \left(\int_{B(o,r/2)} |\nabla \operatorname{Ric}|^a\right)^{\frac{1}{a}} &\leq Cr^{\frac{n}{a}-3}\left(\int_{B(o,r)} |\operatorname{Ric}|^{\frac{n}{2}}\right)^{\frac{2}{n}}\end{aligned}$$

for all $a > 0$*, regardless of the presence of singularities.*

Pf

We try to gain estimates for $L^p(|\nabla \operatorname{Ric}|)$.

$$\begin{aligned}C\left(\int \phi^{l\gamma}|\nabla \operatorname{Ric}|^{k\gamma}\right)^{\frac{1}{\gamma}} &\leq \int \phi^{l-2}|\nabla\phi|^2|\nabla \operatorname{Ric}|^k + \int \phi^l|\nabla \operatorname{Ric}|^{k-2}|\nabla|\nabla \operatorname{Ric}||^2\\ \int \phi^l|\nabla \operatorname{Ric}|^{k-2}|\nabla|\nabla \operatorname{Ric}||^2 &\leq \int \phi^{l-2}|\nabla\phi|^2|\nabla \operatorname{Ric}|^k - \int \phi^l|\nabla \operatorname{Ric}|^{k-2}\langle\nabla \operatorname{Ric}, \triangle\nabla \operatorname{Ric}\rangle.\end{aligned}$$

Using a commutator formula on the last term, we get

$$\begin{aligned}&\int \phi^l|\nabla \operatorname{Ric}|^{k-2}|\nabla|\nabla \operatorname{Ric}||^2\\ &\leq \int \phi^{l-2}|\nabla\phi|^2|\nabla \operatorname{Ric}|^k + \int \phi^l|\nabla \operatorname{Ric}|^k|\operatorname{Rm}|\\ &\quad + l\int \phi^{l-1}|\nabla\phi||\nabla \operatorname{Ric}|^{k-1}|\triangle \operatorname{Ric}| + \int \phi^l|\nabla \operatorname{Ric}|^{k-2}|\triangle \operatorname{Ric}|^2\\ &\quad + (k-2)\int \phi^l|\nabla \operatorname{Ric}|^{k-3}\langle\nabla \operatorname{Ric}, \nabla|\nabla \operatorname{Ric}|\otimes\triangle \operatorname{Ric}\rangle.\end{aligned}$$

Therefore

$$\begin{aligned}&\int \phi^l|\nabla \operatorname{Ric}|^{k-2}|\nabla|\nabla \operatorname{Ric}||^2\\ &\leq C\int \phi^{l-2}|\nabla\phi|^2|\nabla \operatorname{Ric}|^k + C\int \phi^l|\nabla \operatorname{Ric}|^k|\operatorname{Rm}|\\ &\quad + C\int \phi^l|\nabla \operatorname{Ric}|^{k-2}|\triangle \operatorname{Ric}|^2\end{aligned}$$

and so

$$C\left(\int \phi^{l\gamma}|\nabla \operatorname{Ric}|^{k\gamma}\right)^{\frac{1}{\gamma}} \leq \int \phi^{l-2}|\nabla\phi|^2|\nabla \operatorname{Ric}|^k + \int \phi^l|\nabla \operatorname{Ric}|^{k-2}|\triangle \operatorname{Ric}|^2$$

$$\begin{aligned}C\left(\int \phi^{l\gamma}|\nabla \operatorname{Ric}|^{k\gamma}\right)^{\frac{1}{\gamma}} &\leq \int \phi^{l-2}|\nabla\phi|^2|\nabla \operatorname{Ric}|^k\\ &\quad + \int \phi^l|\nabla \operatorname{Ric}|^{k-2}|\operatorname{Ric}|^2|\operatorname{Rm}|^2 + \int \phi^l|\nabla \operatorname{Ric}|^{k-2}|\nabla X|^2 \qquad (46)\end{aligned}$$

To continue we must estimate $\int |\operatorname{Rm}|^a$ locally:

$$\left(\int \phi^{k\gamma}|\operatorname{Rm}|^{k\gamma}\right)^{\frac{1}{\gamma}} \leq \int \phi^{l-2}|\nabla\phi|^2|\operatorname{Rm}|^k + \int \phi^l|\operatorname{Rm}|^{k-2}|\nabla|\operatorname{Rm}||^2$$

$$\begin{aligned} &C\int \phi^l|\operatorname{Rm}|^{k-2}|\nabla|\operatorname{Rm}||^2 \\ &\leq \int \phi^{l-2}|\nabla\phi|^2|\operatorname{Rm}|^k - \int \phi^l|\operatorname{Rm}|^{k-2}\langle \operatorname{Rm}, \triangle \operatorname{Rm}\rangle \\ &C\int \phi^l|\operatorname{Rm}|^{k-2}|\nabla|\operatorname{Rm}||^2 \\ &\leq \int \phi^{l-2}|\nabla\phi|^2|\operatorname{Rm}|^k + \int \phi^l|\operatorname{Rm}|^{k+1} - \int \phi^l|\operatorname{Rm}|^{k-2}\left\langle \operatorname{Rm}, \nabla^2 \operatorname{Ric}\right\rangle \end{aligned}$$

$$\begin{aligned} &C\int \phi^l|\operatorname{Rm}|^{k-2}|\nabla|\operatorname{Rm}||^2 \\ &\leq \int \phi^{l-2}|\nabla\phi|^2|\operatorname{Rm}|^k + \int \phi^l|\operatorname{Rm}|^{k+1} + \int \phi^l|\operatorname{Rm}|^{k-2}\langle \operatorname{Rm}, \nabla\phi\otimes\nabla \operatorname{Ric}\rangle \\ &\quad + \int \phi^l|\operatorname{Rm}|^{k-2}\langle \operatorname{Rm}, \nabla|\operatorname{Rm}|\otimes\nabla \operatorname{Ric}\rangle + \int \phi^l|\operatorname{Rm}|^{k-2}\langle \nabla \operatorname{Ric}, \nabla \operatorname{Ric}\rangle \end{aligned}$$

$$\begin{aligned} &C\int \phi^l|\operatorname{Rm}|^{k-2}|\nabla|\operatorname{Rm}||^2 \\ &\leq \int \phi^{l-2}|\nabla\phi|^2|\operatorname{Rm}|^k + \int \phi^l|\operatorname{Rm}|^{k+1} + \int \phi^l|\operatorname{Rm}|^{k-2}|\nabla \operatorname{Ric}|^2 \end{aligned}$$

Doing the same integration-by-parts on the last term we get finally

$$\begin{aligned} &C\int \phi^l|\operatorname{Rm}|^{k-2}|\nabla|\operatorname{Rm}||^2 \\ &\leq \int \phi^{l-2}|\nabla\phi|^2|\operatorname{Rm}|^k + \int \phi^l|\operatorname{Rm}|^{k+1} + \int \phi^l|\operatorname{Rm}|^{k-2}\langle \operatorname{Ric}, \triangle \operatorname{Ric}\rangle \end{aligned}$$

Altogether therefore,

$$\left(\int \phi^{k\gamma}|\operatorname{Rm}|^{k\gamma}\right)^{\frac{1}{\gamma}} \leq \int \phi^{l-2}|\nabla\phi|^2|\operatorname{Rm}|^k + \int \phi^l|\operatorname{Rm}|^{k+1} + \int \phi^l|\operatorname{Rm}|^{k-1}|\nabla X|$$

Using Hölder's inequality and Lemma 4.8 on the last term, we get that

$$\left(\int \phi^{k\gamma}|\operatorname{Rm}|^{k\gamma}\right)^{\frac{1}{\gamma}} \leq \int \phi^{l-2}|\nabla\phi|^2|\operatorname{Rm}|^k,$$

which, using Lemma 4.11, holds across singularities for all k. Iterating this gives

$$\left(\int_{B(o,r)} |\operatorname{Rm}|^a\right)^{\frac{1}{a}} \leq r^{\frac{n}{a}-2}\left(\int_{B(o,r/2)} |\operatorname{Rm}|^{\frac{n}{2}}\right)^{\frac{2}{n}}.$$

Returning to (46), we get

$$
\begin{aligned}
C&\left(\int \phi^{l\gamma}|\nabla \operatorname{Ric}|^{k\gamma}\right)^{\frac{1}{\gamma}} \\
&\leq \int \phi^{l-2}|\nabla\phi|^2|\nabla \operatorname{Ric}|^k + \left(\int \phi^{l}|\nabla \operatorname{Ric}|^{k\gamma}\right)^{\frac{k-2}{k}\frac{1}{\gamma}} \left(\int |\nabla X|^{\frac{nk}{n+k-2}}\right)^{\frac{2(k+n-2)}{kn}} \\
&\quad + \left(\int \phi^{l\gamma}|\nabla \operatorname{Ric}|^{k\gamma}\right)^{\frac{k-2}{k}\frac{1}{\gamma}} \left(\int |\operatorname{Ric}|^{k\gamma}\right)^{\frac{2}{k}\frac{1}{\gamma}} \left(\int |\operatorname{Rm}|^{n}\right)^{\frac{2}{n}}
\end{aligned}
$$

knowing what we do about $L^n(|\operatorname{Rm}|)$, $L^p(|\operatorname{Ric}|)$, and $L^p(|\nabla X|)$, we iterate to get

$$
C\left(\int_{B(o,r/2)} |\nabla \operatorname{Ric}|^a\right)^{\frac{1}{a}} \quad \leq \quad r^{\frac{n}{a}-\frac{n}{2}}\left(\int_{B(o,r)} |\nabla \operatorname{Ric}|^2\right)^{\frac{1}{2}} + r^{\frac{n}{a}-3}\left(\int_{B(o,r)} |\operatorname{Ric}|^{\frac{n}{2}}\right)^{\frac{2}{n}}.
$$

We can easily estimate $\int |\nabla \operatorname{Ric}|^2$ in terms of $|\operatorname{Ric}|$, so we can get

$$
\left(\int_{B(o,r/2)} |\nabla \operatorname{Ric}|^a\right)^{\frac{1}{a}} \quad \leq \quad Cr^{\frac{n}{a}-3}\left(\int_{B(o,r)} |\nabla \operatorname{Ric}|^{\frac{n}{2}}\right)^{\frac{2}{n}}
$$

□

4.4 Removing curvature singularities, $n = 4$

In this section we prove that for some $s > 0$, $|\operatorname{Rm}| = O(r^{-2+s})$ in dimension 4, where r indicates distance to a singularity. Although we do not get specific bounds of the sort in Theorem 4.1, this result is enough to prove the full removable singularity theorem in Section 5. Some parts of the argument are glossed over here; a complete argument can be found in the thesis of the second author [Web1].

In dimension 4 the situation is unfortunately less straightforward than in higher dimensions. Roughly speaking our coupled elliptic system has the form $\triangle u \geq -fu - g$, where $u \geq 0$ is some curvature quantity. In dimension 4 the hypothesis that $f, g \in L^{\frac{n}{2}}$ is insufficient for a purely analytical argument to remove a point singularity. The counterexample is $u = -r^{-2}(\log r)^{-1}$, for which Sibner's lemma also fails.

We look again to the geometry of our manifolds to provide us additional input. Uhlenbeck's 1982 paper on Yang-Mills connections introduced what has become a standard technique here, which we briefly review. After a choice of gauge (local coordinates) the connection can be written $D = d+A$, with A being an $\mathfrak{so}(n)$-valued 1-form, and the curvature F, by definition just $D \circ D$, can be written $F = DA - \frac{1}{2}[A, A]$. Uhlenbeck used the implicit function theorem to show that in the annulus, if the metric is almost flat and the gauge is chosen so A is small, the gauge can be slightly modified to make $D^*A = 0$. A gauge in which this holds is called a Hodge gauge. To get better control on $L^2(|F|)$, one estimates on the annulus Ω,

$$\begin{aligned}\int_\Omega |F|^2 &= \int_\Omega \left\langle DA - \frac{1}{2}[A, A], F \right\rangle \\ &= \int_\Omega \langle a, D^*F \rangle + C \int_\Omega |A|^2|F| + \textit{Boundary Terms}.\end{aligned}$$

Working in a Hodge gauge has the advantage of making certain estimates involving A possible; for instance the $\int |A|^2|F|$ term can be estimated. The D^*F term is ordinarily uncontrollable, but whatever advantage one can squeeze out here may improve the estimate for $\int |F|^2$. For Yang-Mills connections $D^*F = 0$ by definition; this is also true in the Einstein case. In general the second Bianchi identity gives only that D^*F is a combination of $\nabla \operatorname{Ric}$ terms, so in principle better control over $L^2(|F|)$ can come from better control over $L^p(|\nabla \operatorname{Ric}|)$. This was essentially the method of [TV1], where they were able to get improved estimates for $|\operatorname{Ric}|$, and then for $|\nabla \operatorname{Ric}|$. Assuming that a good L^p estimate for $|\nabla \operatorname{Ric}|$ is somehow achieved, one gets $L^2(|F|)$ estimates on the full punctured disk by estimating on successively smaller annuli, piecing together the boundary terms, and showing that the residue (the inner boundary term on the shrinking annuli) vanishes. For details, see [Uhl], [Tia1], [TV1].

In the case most similar to ours, Theorem 6.4 of [TV1], better control on $L^2(|\nabla \operatorname{Ric}|)$ is achieved using an *improved Kato inequality* for the Ricci curvature, which yields an improved elliptic inequality. Their inequality relied on the Kähler metric having constant scalar curvature, so their particular estimates are unavailable to us. In the proof below we essentially take advantage of the holomorphicity of X to recover some information about the irreducible $U(n)$ decomposition of derivatives of curvature tensor. However we only partially recover an improved Kato inequality and more effort is needed to achieve something useful. Although our method of proof is standard, we run through it again because the value of the constant actually turns out to be important.

Assume V is a complex vector space. Let $\mathcal{A}$ be the space of tensors $A_{i\bar{j}kl}$ of type $V \otimes \overline{V} \otimes V \otimes V$ such that A is trace-free in the first two positions and symmetric in the first and third positions; that is $\Sigma_s A_{s\bar{s}kl} = 0$ and $A_{i\bar{j}kl} = A_{k\bar{j}il}$. Let $\mathcal{B}$ be the space of tensors $B_{i\bar{j}k}$ of type $V \otimes \overline{V} \otimes V$ that are trace-free in the first two positions.

Lemma 4.13 *Assume V has complex dimension m. Let $\langle\, , \rangle : \mathcal{A} \times \overline{\mathcal{B}} \to V$ denote the trace in the first three positions. Then $|\langle a, B \rangle|^2 \leq \frac{m-1}{2m}|A|^2|B|^2$ when $B_{i\bar{s}s} \neq 0$.*

Pf

Restricting ourselves to tensors of unit norm, and using Lagrange's multiplier method, one finds

$$\left\langle\left\langle \widetilde{A}, B\right\rangle, \langle a, B\rangle\right\rangle = a\left\langle a, \widetilde{A}\right\rangle \tag{47}$$

$$\left\langle\left\langle a, \widetilde{B}\right\rangle, \langle a, B\rangle\right\rangle = b\left\langle B, \widetilde{B}\right\rangle, \tag{48}$$

for $\widetilde{A} \in \mathcal{A}$, $\widetilde{B} \in \mathcal{B}$ arbitrary. Clearly $a = b = |\langle a, B\rangle|^2$. Letting λ be the vector $\lambda = \frac{1}{|\langle a,B\rangle|^2}\langle a, B\rangle$, (47) and (48) can be written

$$\begin{aligned}\left\langle \widetilde{A}, B\otimes\lambda\right\rangle &= \left\langle \widetilde{A}, A\right\rangle \\ \left\langle \langle a,\lambda\rangle, \widetilde{B}\right\rangle &= \left\langle B, \widetilde{B}\right\rangle.\end{aligned}$$

This means that, with π_1 the projection onto $\mathcal{A}$ and π_2 the projection onto $\mathcal{B}$, B satisfies

$$B = \pi_2\langle \pi_1(B\otimes\lambda), \lambda\rangle. \tag{49}$$

For arbitrary $\widetilde{A} \in \mathcal{A}$, $\widetilde{B} \in \mathcal{B}$,

$$\begin{aligned}\pi_1(\widetilde{A}) &= \frac{1}{2}\left(\widetilde{A}_{i\bar{j}kl} + \widetilde{A}_{k\bar{j}il}\right) - \frac{1}{2m}\delta_{i\bar{j}}\left(\widetilde{A}_{s\bar{s}kl} + \widetilde{A}_{k\bar{s}sl}\right) \\ \pi_2(\widetilde{B}) &= \widetilde{B}_{i\bar{j}k} - \frac{1}{m}\delta_{i\bar{j}}\widetilde{B}_{s\bar{s}k}.\end{aligned}$$

Then we compute

$$\pi_2\langle\pi_1(B\otimes\lambda),\lambda\rangle = \frac{1}{2}\left(B_{i\bar{j}k} + B_{k\bar{j}i}\right)|\lambda|^2 - \frac{1}{2m}\delta_{i\bar{j}}B_{k\bar{s}s}|\lambda|^2.$$

Tracing both sides of (49) in j, k, gives

$$B_{i\bar{s}s}\left(2 - |\lambda|^2 + \frac{1}{m}|\lambda|^2\right) = 0, \tag{50}$$

so either B is trace-free in the second two variables, or $|\lambda|^2 = \frac{2m}{m-1}$. □

Proposition 4.14 (Improved Kato Inequality) *Let M be an extremal Kähler manifold of complex dimension m and of nonconstant scalar curvature. Denote by $E_{i\bar{j}} = \mathrm{Ric}_{i\bar{j}} - \frac{1}{m}h_{i\bar{j}}R$ the trace-free Ricci tensor. Then*

$$2|\nabla|\nabla E||^2 \le \frac{m-1}{2m}|\nabla^2 E|^2 + |\overline{\nabla}\nabla E|^2, \tag{51}$$

where we denote $\nabla^2 E = E_{i\bar{j},kl}$ and $\overline{\nabla}\nabla E = E_{i\bar{j},k\bar{l}}$.

Pf

Adopting the notation from above, we have $\nabla^2 E \in \mathcal{A}$ and $\nabla E \in \mathcal{B}$. Therefore

$$|\left\langle \nabla^2 E, \nabla E \right\rangle|^2 \leq \frac{m-1}{2m} |\nabla^2 E|^2 |\nabla E|^2 \tag{52}$$

The result follows from the identity

$$\nabla |\nabla E|^2 \quad = \quad \left\langle \nabla^2 E, \nabla E \right\rangle + \left\langle \nabla E, \overline{\nabla} \nabla E \right\rangle .$$

□

Lemma 4.15 (Improved elliptic inequality) *If* $|\alpha - 1 - \delta| < \frac{\sqrt{32}}{5}$, *then*

$$\begin{aligned}
&\int \phi^2 |\nabla E|^\alpha \triangle |\nabla E|^{1-\delta} \\
&\quad \geq -\frac{1}{2}(1-\delta)\delta C \int |\nabla \phi|^2 |\nabla E|^{1-\delta+\alpha} - \frac{1}{2}(1-\delta)\delta C \int \phi^2 |\operatorname{Ric}| |\nabla E|^{1-\delta+\alpha} \\
&\qquad + \frac{1}{2}(1-\delta) \int \phi^2 |\nabla E|^{-1-\delta+\alpha} \left(\langle \triangle \nabla E, \nabla E \rangle + \left\langle \nabla E, \overline{\triangle} \nabla E \right\rangle \right).
\end{aligned}$$

Pf

Using the improved Kato inequality and, and setting $\eta = \frac{m-1}{2m}$, we get

$$\begin{aligned}
\triangle |\nabla E|^{1-\delta} \quad &= \quad \frac{1}{2}(1-\delta)|\nabla E|^{-1-\delta} \left((1+\delta)\eta |\nabla^2 E|^2 - \delta |\overline{\nabla} \nabla E|^2 \right) \\
&\qquad + \frac{1}{2}(1-\delta)|\nabla E|^{-1-\delta} \left(\langle \triangle \nabla E, \nabla E \rangle + \left\langle \nabla E, \overline{\triangle} \nabla E \right\rangle \right)
\end{aligned} \tag{53}$$

We want $(1+\delta)\eta|\nabla^2 E|^2 - \delta|\overline{\nabla}\nabla E|^2 \geq 0$, though this does not seem possible in the pointwise sense. We will have better luck after integration however. Using integration by parts and a commutator formula, we get

$$\begin{aligned}
&\int \phi^2 |\nabla E|^\beta |\overline{\nabla} \nabla E|^2 \\
&\quad \leq \int \phi^2 |\nabla E|^\beta |\nabla^2 E|^2 \\
&\qquad + 2 \int \phi |\nabla \phi| |\nabla E|^{1+\beta} |\nabla^2 E| + 2 \int \phi |\nabla \phi| |\nabla E|^{1+\beta} |\overline{\nabla} \nabla E| \\
&\qquad + |\beta| \int \phi^2 |\nabla E|^\beta |\nabla^2 E| |\nabla |\nabla E|| + |\beta| \int \phi^2 |\nabla E|^\beta |\overline{\nabla} \nabla E| |\nabla |\nabla E|| \\
&\qquad + 3 \int \phi^2 |\nabla E|^{\beta+2} |\operatorname{Ric}|
\end{aligned}$$

Then with $|\nabla|\nabla E|| \leq \frac{1}{\sqrt{2}}|\nabla^2 E| + \frac{1}{\sqrt{2}}|\overline{\nabla}\nabla E|$ and assuming that $|\beta| < \frac{\sqrt{32}}{5}$, we get

$$\left(1-\frac{5|\beta|}{4\sqrt{2}}\right)\int \phi^2|\nabla E|^\beta|\overline{\nabla}\nabla E|^2$$
$$\leq \left(2+\frac{33|\beta|}{\sqrt{2}}\right)\int \phi^2|\nabla E|^\beta|\nabla^2 E|^2$$
$$+3\int \phi^2|\nabla E|^{\beta+2}|\operatorname{Ric}| + \left(1+\frac{8\sqrt{2}}{|\beta|}\right)\int |\nabla\phi|^2|\nabla E|^{2+\beta}.$$

In fact it is only really necessary that $|\beta| < \sqrt{2}$, but this method does not allow for arbitrary β. With $|\alpha| < \frac{\sqrt{32}}{5} - |1+\delta|$ we therefore get

$$\int \phi^2|\nabla E|^\alpha \triangle|\nabla E|^{1-\delta}$$
$$\geq \frac{1}{2}(1-\delta)\left(1-\eta(1+\delta)-\delta C\right)\int \phi^2|\nabla E|^{-1-\delta+\alpha}|\nabla^2 E|^2$$
$$-\frac{1}{2}(1-\delta)\delta C\int |\nabla\phi|^2|\nabla E|^{1-\delta+\alpha} - \frac{1}{2}(1-\delta)\delta C\int \phi^2|\operatorname{Ric}||\nabla E|^{1-\delta+\alpha}$$
$$+\frac{1}{2}(1-\delta)\int \phi^2|\nabla E|^{-1-\delta+\alpha}\left(\langle\triangle\nabla E,\nabla E\rangle + \langle\nabla E,\overline{\triangle}\nabla E\rangle\right).$$

The first term is positive when δ is sufficiently small. □

The next lemma shows how to use this improved elliptic inequality.

Lemma 4.16 *Assume* $\operatorname{supp}\phi$ *consists of manifold points. There exists an* $\epsilon_0 > 0$ *so that* $\int_{\operatorname{supp}\phi}|\operatorname{Ric}|^2 < \epsilon_0$ *implies*

$$\left(\int \phi^4|\nabla E|^2\right)^{\frac{1}{2}} \leq C\int |\nabla\phi|^2|\nabla E| + C\int \phi^2|\operatorname{Rm}||X| + C\left(\int |\operatorname{Rm}|^2\right)^{\frac{1}{2}}\left(\int \phi^2|\nabla\phi|^2|E|^2\right)^{\frac{1}{2}}$$

<u>*Pf*</u>

Set $u = |\nabla E|^{1-\delta}$ and use the Sobolev inequality to get

$$C\left(\int \phi^4 u^{2\frac{1}{1-\delta}}\right)^{\frac{1}{2}} \leq \int |\nabla\phi|^2 u^{\frac{1}{1-\delta}} - \int \phi^2 u^{\frac{\delta}{1-\delta}}\triangle u$$

Since Lemma 4.15 holds for $\alpha = \delta$ and $\int|\operatorname{Ric}|^2$ is assumed small, we get

$$\left(\int \phi^4|\nabla E|^2\right)^{\frac{1}{2}} \leq C\int |\nabla\phi|^2|\nabla E| + C\int \phi^2|\triangle\nabla E|$$

for $C = C(C_S)$. We'll use

$$\triangle \nabla E \;=\; \nabla \operatorname{Rm} * E \,+\, \operatorname{Rm} * \nabla E \,+\, \operatorname{Rm} * X$$

First note that $\operatorname{Rm} * X \in L^1$, since both are in L^2. Also, $\int \phi^2 |\operatorname{Rm}||\nabla E|$ can be combined into the left side. Altogether,

$$C\left(\int \phi^4 |\nabla E|^2\right)^{\frac{1}{2}} \;\leq\; \int |\nabla \phi|^2 |\nabla E| + \int \phi^2 |\nabla \operatorname{Rm}||E| + \int \phi^2 |\operatorname{Rm}||X|$$

The Sobolev inequality directly gives

$$\left(\int \phi^8 |E|^4\right)^{\frac{1}{2}} \;\leq\; C\int \phi^2 |\nabla \phi|^2 |E|^2 + C\int \phi^4 |\nabla E|^2.$$

Since we are working in the smooth case, we may also our previous result that

$$\left(\int |\nabla \operatorname{Rm}|^{\frac{4}{3}}\right)^{\frac{3}{4}} \;\leq\; C\left(\int |\operatorname{Rm}|^2\right)^{\frac{1}{2}}$$

assuming the domain of the second integral is somewhat larger than the domain of the first. The result immediately follows. □

As we now show, this lemma gives us just enough to conclude that $|\nabla \operatorname{Ric}| \in L^{\frac{4}{3}}$. Due to Theorem 4.1 already have $|\nabla \operatorname{Ric}| \in L^p$ for all $p < \frac{4}{3}$. For a similar argument, see the proof of Theorem 5.8 of [BKN].

Lemma 4.17 *Its holds that $|\nabla E| \in L^{\frac{4}{3}}$, and in fact given any $\beta > 1$,*

$$\int_{B(o,\rho)} |\nabla \operatorname{Ric}|^{\frac{4}{3}} \;\leq\; C\int_{B(o,\beta\rho)-B(o,\rho)} |\nabla \operatorname{Ric}|^{\frac{4}{3}} + C\int_{B(o,\beta\rho)} |\operatorname{Rm}|^2,$$

$C = C(C_S, \beta)$, despite the possible presence of singularities.

Pf

Choosing any $k \in (1,2]$, the previous lemma and Hölder's inequality gives

$$\begin{aligned}\int \phi^{2k}|\nabla E|^k \;\leq\;& C(\operatorname{Vol}\operatorname{supp}\phi)^{1-\frac{k}{2}}\left(\int |\nabla\phi|^{\frac{2k}{k-1}}\right)^{k-1}\int_{\operatorname{supp}\nabla\phi}|\nabla E|^k\\ &+C(\operatorname{Vol}\operatorname{supp}\phi)^{1-\frac{k}{2}}\left(\int |\nabla\phi|^2\right)^{\frac{k}{2}}\left(\max_{\operatorname{supp}|\nabla\phi|}|E|\right)^k\left(\int|\operatorname{Rm}|^2\right)^{\frac{k}{2}}\\ &+C(\operatorname{Vol}\operatorname{supp}\phi)^{\frac{1}{k}-\frac{1}{2}}\int\phi^2|\operatorname{Rm}||X|.\end{aligned}$$

This holds assuming no singularity lies in $\operatorname{supp}\phi$. Now we let the ϕ be test functions with support everywhere except for small balls around the singularities. For simplicity we can assume there is a single singularity; if there multiple singularities this method requires that the test functions must close in around all of them simultaneously. Choose some number $\beta > 1$ and let ϕ_i be a sequence test function with $\mathrm{supp}\,\phi_i \cap B(o, \beta^{-i-1}) = \varnothing$, with $\phi_i \equiv 1$ in $M - B(o, \beta^{-i})$, and with $|\nabla \phi_i| \leq 2\beta^{i+1}$.

Set $A_i = \int_{M-B(o,\beta^{-i})} |\nabla E|^k$. Then $\int_{\mathrm{supp}\,|\nabla\phi|} |\nabla E|^k \; = \; A_{i+1} - A_i$. With $|E| = o(r^{-2})$ near singularities, inequality (54) takes the form

$$\begin{aligned} A_i &\leq C\,(A_{i+1} - A_i) \,+\, C\beta^{-i(4-3k)} \,+\, C\beta^{-i(\frac{4}{k}-2)} \\ A_i &\leq \frac{C}{1+C} A_{i+1} \,+\, \frac{C}{1+C}\beta^{-i(4-3k)} \,+\, \frac{C}{1+C}\beta^{-i(\frac{4}{k}-2)}. \end{aligned}$$

Iterating, we get

$$\begin{aligned} A_i \;\leq\; & \left(\frac{C}{1+C}\right)^N A_{i+N} \\ & + \frac{C}{1+C}\beta^{-i(4-3k)} \left(1 + \frac{C}{1+C}\beta^{-(4-3k)} \,+\, \ldots \,+\, \left(\frac{C}{1+C}\beta^{-(4-3k)}\right)^{N-1}\right) \\ & + \frac{C}{1+C}\beta^{-i(\frac{4}{k}-2)} \left(1 + \frac{C}{1+C}\beta^{-(\frac{4}{k}-2)} \,+\, \ldots \,+\, \left(\frac{C}{1+C}\beta^{-(\frac{4}{k}-2)}\right)^{N-1}\right) \end{aligned}$$

An advantage is possible in the boundary case where $k = \frac{4}{3}$. In this case clearly the last two terms are bounded independently of N. In the case $k = \frac{4}{3}$, we know that A_{i+N} grows slower than any power of β^N; therefore as $N \to \infty$ the first term vanishes. Thus

$$A_i \;\leq\; 1 + C,$$

which is a bound independent of i. Letting $i \to \infty$ yields the theorem.

Now with $|\nabla E| \in L^{\frac{4}{3}}$, one easily gets that

$$\begin{aligned} \left(\int \phi^4 |\nabla E|^2\right)^{\frac{1}{2}} \;\leq\; & C\int |\nabla\phi|^2|\nabla E| \,+\, C\int \phi^2 |\,\mathrm{Rm}\,||X| \\ & + C\left(\int |\,\mathrm{Rm}\,|^2\right)^{\frac{1}{2}} \left(\int \phi^2|\nabla\phi|^2|E|^2\right)^{\frac{1}{2}} \end{aligned}$$

holds regardless of singularities. With $|\nabla\,\mathrm{Ric}\,|^2 \leq |\nabla E|^2 + \frac{1}{n}|X|^2$ and $|\nabla E| \leq C(n)|\nabla\,\mathrm{Ric}\,|$, we get

$$\begin{aligned} \left(\int \phi^4 |\nabla\,\mathrm{Ric}\,|^2\right)^{\frac{1}{2}} \;\leq\; & C\int |\nabla\phi|^2|\nabla\,\mathrm{Ric}\,| \,+\, C\int \phi^2 |\,\mathrm{Rm}\,||X| \\ & + Cr^{-1}\int |\,\mathrm{Rm}\,|^2 \,+\, C\left(\int \phi^2|X|^2\right)^{\frac{1}{2}} \end{aligned}$$

Using Hölder's inequality and that $|X| = o(r^{-3})$ points near singularities,

$$\left(\int_{B(o,r/2)} |\nabla \operatorname{Ric}|^{\frac{4}{3}}\right)^{\frac{3}{4}} \leq C\left(\int_{B(o,r)} |\nabla \operatorname{Ric}|^{\frac{4}{3}}\right)^{\frac{3}{4}} + C\int_{B(o,r)} |\operatorname{Rm}|^2.$$

Using that $\int |\operatorname{Rm}|^2$ is presumed small, we get the lemma. $\square$

Lemma 4.18 (Uhlenbeck's method) *Assume $|F| = o(r^{-2})$ near singularities and that $|\nabla \operatorname{Ric}| \in L^{\frac{4}{3}}$. If o is a singuarity, we can choose ρ small enough and and β large enough so that*

$$\int_{B(o,\rho)} |F|^2 \leq \frac{1}{2}\int_{B(o,\rho\beta)} |F|^2 + C\int_{B(o,\rho\beta)} |\nabla \operatorname{Ric}|^{\frac{4}{3}}.$$

where C is some universal constant.

<u>*Pf*</u>

We will not present Uhlenbeck's argument in its entirety here, but our use of it will be unique enough that we must repeat some of the proof. Uhlenbeck first proves that a gauge can be found on the annulus so that $D^*A = 0$, where A is the connection 1-form. One of the main advantages of computing in this special gauge is that integral norms of $|A|$ are bounded in terms of those of $|F|$ (see [Tia1], [Uhl]). In fact given a domain Ω, we can get

$$\int_\Omega |A|^2 \leq C\int_\Omega |F|^2 \tag{54}$$

where $C = C(\Omega)$. Our computation is similar to those in [Uhl], [Tia1], and [TV1], but we use test function methods rather than try to control the boundary terms. We will use the second Bianchi identity $D^*F = D\operatorname{Ric}$. We get

$$\begin{aligned}
\int \phi^2|F|^2 &= \int \phi^2 \left\langle DA - \frac{1}{2}[A,A], F\right\rangle \qquad (55)\\
&= \int \phi^2 \langle DA, F\rangle - \frac{1}{2}\int \phi^2 \langle [A,A], F\rangle \\
&= -2\int \phi \langle \nabla\phi \otimes A, F\rangle - \int \phi^2 \langle a, D^*F\rangle - \frac{1}{2}\int \phi^2 \langle [A,A], F\rangle \\
&= -2\int \phi \langle \nabla\phi \otimes A, F\rangle - \int \phi^2 \langle a, D\operatorname{Ric}\rangle - \frac{1}{2}\int \phi^2 \langle [A,A], F\rangle
\end{aligned}$$

In a Hodge gauge it is possible to estimate $\int |A|^4$ in terms of $\int |F|^2$. The Sobolev inequality gives

$$\left(\int |A|^4\right)^{\frac{1}{2}} \;\le\; C_S\int |DA|^2 \,+\, (\mathrm{Vol}\,\mathrm{supp}\,\phi)^{-\frac{1}{2}}\int |A|^2$$
$$\le\; C_S\int |F|^2 \,+\, \left(C_S \sup_{\mathrm{supp}\,\phi}|A|^2 + (\mathrm{Vol}\,\mathrm{supp}\,\phi)^{-\frac{1}{2}}\right)\int |A|^2$$
$$\le\; C\int |F|^2 \tag{56}$$

where we have used (54). Here C depends on the Sobolev constant and on $\sup_{\mathrm{supp}\,\phi}|F|$. We now get from (55),

$$\int \phi^2|F|^2 \;\le\; c\int |\nabla\phi|^2|A|^2 \,+\, c\int \phi^2|D\,\mathrm{Ric}\,|^{\frac{4}{3}} \tag{57}$$

We want to estimate $\int |A|^2$ back in terms of $\int |F|^2$, but we need to control the coefficient. We get becomes

$$\int \phi^2|F|^2 \;\le\; c\left(\int |\nabla\phi|^4\right)^{\frac{1}{2}}\left(\int_{\mathrm{supp}\,\phi}|A|^4\right)^{\frac{1}{2}} \,+\, c\int \phi^2|D\,\mathrm{Ric}\,|^{\frac{4}{3}}$$
$$\le\; c\left(\int |\nabla\phi|^4\right)^{\frac{1}{2}}\int |F|^2 \,+\, c\int \phi^2|D\,\mathrm{Ric}\,|^{\frac{4}{3}}$$

Assuming ϕ is defined in the annulus $B(o,1)-B(o,\beta^{-1})$, we can make $\int |\nabla\phi|^4$ very small by making β large; in fact we can make $\int |\nabla\phi|^4 \backsim (\log\beta)^{-3}$. This done, we get

$$\int \phi^2|F|^2 \;\le\; \epsilon\int_{\mathrm{supp}\,\phi}|F|^2 \,+\, c\int \phi^2|D\,\mathrm{Ric}\,|^{\frac{4}{3}}$$

We can choose β large enough so that $\epsilon < \frac{1}{4}$. A significant subtlety is that as the annulus goes to zero $B(o,1)-B(o,\beta^{-1})$ degenerates to a punctured disk, the estimate (57) does not degenerate. It is possible to prove this with a modification of the argument on pg. 129 of [Tia1]; see [Web1] for the details.

We now piecing together successively smaller annuli, in order to close in around the singularity. Let ϕ_i be a test function with $\phi_i \equiv 1$ in $B(o,\beta^{-i-1}) - B(o,\beta^{-i-2})$, $\phi_i \equiv 0$ in $B(o,\beta^{-i-3})$ and outside $B(o,\beta^{-i})$, and also $\int |\nabla\phi_i|^4 \le C(\log\beta)^{-3}$. Then our inequality reads

$$\int_{B(o,\beta^{-i-1})\cap B(o,\beta^{-i-2})}|F|^2$$
$$\le \epsilon\int_{B(o,\beta^{-i-2})-B(o,\beta^{-i-3})}|F|^2$$
$$+\,\epsilon\int_{B(o,\beta^{-i})-B(o,\beta^{-i-1})}|F|^2$$
$$+\,c\int_{B(o,\beta^{-i})-B(o,\beta^{-i-3})}|D\,\mathrm{Ric}\,|^{\frac{4}{3}}$$

Now summing both sides from $i = N$ to ∞ gives

$$\int_{B(o,\beta^{-N-1})} |F|^2 \le \frac{1}{2}\int_{B(o,\beta^{-N})} |F|^2 + 3c\int_{B(o,\beta^{-N})} |D\,\mathrm{Ric}\,|^{\frac{4}{3}}.$$

□

Proposition 4.19 *Assuming Lemmas 4.17 and 4.18, we have that $|F| \in L^p$ for some $p > 2$.*

<u>*Pf*</u>

Propositions 4.17 and 4.18 now give

$$\int_{B(o,\rho)} |F|^2 \le \int_{B(o,\rho\beta)-B(o,\rho)} |F|^2 + 3c\int_{B(o,\rho\beta)} |D\,\mathrm{Ric}\,|^{\frac{4}{3}}.$$

$$\int_{B(o,\rho)} |\nabla\,\mathrm{Ric}\,|^{\frac{4}{3}} \le C\int_{B(o,\rho\beta)-B(o,\rho)} |\nabla\,\mathrm{Ric}\,|^{\frac{4}{3}} + C\int_{B(o,\rho\beta)} |F|^2.$$

Setting

$$A_i = \int_{B(o,\rho\beta^{-i})} |F|^2 \qquad B_i = \int_{B(o,\rho\beta^{-i})} |\nabla\,\mathrm{Ric}\,|^{\frac{4}{3}}.$$

we have

$$\begin{aligned} A_i &= C(A_{i-1} - A_i) + CB_{i-1} \\ B_i &= C(B_{i-1} - B_i) + CA_{i-1} \end{aligned}$$

It is possible to iterate these to get

$$A_i = \left(\frac{C}{1+C}\right)^i (A_0 + B_0).$$

Thus choosing $s > 1$ so that $\beta^{-s} = \frac{C}{1+C}$ we get

$$\int_{B(o,\rho\beta^{-i})} |F|^2 \le C'\beta^{-si}.$$

This proves the existence of an $s > 0$ so that $\int_{B(o,r)} |F|^2 = O(r^s)$. Using elliptic regularity (Theorem 4.6) we get that $|\,\mathrm{Rm}\,| = O(r^{-2+s})$ near singularities. Therefore $|\,\mathrm{Rm}\,| \in L^p$ for any $p < \frac{4}{2-s}$. □

Theorem 4.20 *Assume g is an extremal Kähler metric on a Riemannian manifold-with-singularities of dimension 4. When $a > 2$, there exists $\epsilon_0 = \epsilon_0(C_S, a)$ and $C = C(C_S, a)$ so that*

$$\int_{B(o,r)} |\,\mathrm{Rm}\,|^2 \le \epsilon_0$$

implies

$$\left(\int_{B(o,r/2)} |\,\mathrm{Rm}\,|^a\right)^{\frac{1}{a}} \;\leq\; Cr^{-2+\frac{4}{a}}\left(\int_{B(o,r)} |\,\mathrm{Rm}\,|^2\right)^{\frac{1}{2}}. \qquad (58)$$

<u>*Pf*</u>

Now that we know $|\,\mathrm{Rm}\,| \in L^p$ for some $p > 2$, we can use Sibner's Lemma and Proposition 3.5, and repeat the proof of Proposition 4.12. Actually, in the case $n = 4$ Proposition 3.5 is not quite strong enough as stated. Referring to the notation from the statement of 3.5, it is required that $q > 2$. This can be changed to allow equality however. First one makes the following change to the statement of Lemma 3.4: Given $k > \frac{1}{2}\frac{n}{n-2}$, if $u^k \in L^2$ then given any $l \leq k$ it holds that $\nabla u^l \in L^2_{loc}$. □

We would also like to point out that our improved Kato inequality is sufficient for proving an improved curvature decay rate at infinity. That is

$$|\,\mathrm{Rm}\,| \;=\; O(r^{-2-s})$$

as $r \to \infty$, for some $s > 0$. This can be proven using Uhlenbeck's method, essentially just by taking annular regions extending out to infinity rather than in toward a singularity. But considering our rather bulky use of the improved Kato inequality, it is unlikely our method will allow computation of the optimal decay rate.

5 Weak Compactness

In this section we assume our manifolds satisfy a local volume growth upper bound, which is a significant assumption without global, pointwise lower bounds on the Ricci curvature. Following [TV2], we can use the convergence result proved here to turn around and actually *prove* the volume growth assumption, which is done by scaling the manifolds so the local growth condition does hold, and then applying the results of this section. Essentially the possibility of large local volume ratios is counterbalanced by the freedom, in the following argument, to let diameters be as large as desired or even infinite.

We shall adopt the following definition of asymptotically locally Euclidean manifolds: a complete manifold will be called ALE if there exists a compact set $K \subset M$ so that each component of $M - K$ is diffeomorphic to $(\mathbb{R}^n - B)/\Gamma$ for some ball $B \in \mathbb{R}^n$ and some subgroup $\Gamma \subset SO(n)$ (depending on the end), and so that under this identification, the metric components satisfy

$$g_{ij} \;=\; \delta_{ij} \,+\, o(1)$$
$$\partial^k(g_{ij}) \;=\; o(r^{-k}),$$

where ∂^k indicates any partial derivative of order k. In [TV1] for instance, such a manifold is called ALE of order 0.

In this section, we assume $\{(M_\alpha, g_\alpha, x_\alpha)\}_{\alpha\in A}$ is a family of compact, pointed n-dimensional Riemannian manifolds that satisfy

i) Upper bounds on energy: $\int_{M_\alpha} |\operatorname{Rm}|^{\frac{n}{2}} \leq \Lambda$

ii) Lower bounds on volume: $\operatorname{Vol}_{g_\alpha} M_\alpha \geq \nu$

iii) Weak regularity: $\int_{B_r} |\operatorname{Rm}|^{\frac{n}{2}} < \epsilon_0 \;\Rightarrow\; \sup_{B_{r/2}} |\nabla^p \operatorname{Rm}| \leq C r^{-p-2} \left(\int_{B_r} |\operatorname{Rm}|^{\frac{n}{2}}\right)^{\frac{2}{n}}$

iv) Bounded Sobolev constants $C_M < C_S$.

v) Upper bound on local volume growth: $\operatorname{Vol}_{g_\alpha} B(p,r) \leq \overline{v} r^n$ for $0 \leq r \leq 1$

Proposition 5.1 *Let $\{(M_\alpha, g_\alpha, x_\alpha)\}_{\alpha\in A}$ be a family of pointed, compact Riemannian manifolds that satisfy the above conditions. Then a subsequence $\{(M_i, g_i, x_i)\}_{i=i}^\infty$ converges in the pointed Gromov-Hausdorff topology to a complete pointed Riemannian manifold-with-singularities $(M_\infty, g_\infty, x_\infty)$ with at most Λ/ϵ_0 singularities. If M_∞ is noncompact, it is ALE.*

<u>*Pf*</u>

Similar arguments appear frequently in the literature, so we briefly describe the main steps. Choose a small radius $r > 0$. Let $K \subset M$ be the (compact) set of points $p \in M$ where $\int_{B(p,r)} |\operatorname{Rm}|^{\frac{n}{2}} \geq \epsilon_0$. Cover K by balls $B(p_i, 2r)$ such that the $B(p_i, r)$ are disjoint; there can be no more than Λ/ϵ_0 balls in such a covering. Set $\Omega_{i,r,R} = \left(M_i - \bigcup_j B(p_j, 2r)\right) \cap B(x_i, R)$. Notice that when r is small enough, the local volume growth bounds give $\operatorname{Vol}\Omega_{i,r,R} \geq \operatorname{Vol}_{g_i} B(x_i, R) - \epsilon$.

On $\Omega_{i,r,R}$ we have $|\nabla^k \operatorname{Rm}| \leq C r^{-k-2}$. The lower bound on volume growth together with the curvature estimate imply the Cheeger lemma ([Che]) which gives injectivity radius bounds. Therefore we can take a pointed limit along a subsequence of the sets $\Omega_{i,r,R}$ to get a smooth limiting manifold-with-boundary $\Omega_{\infty,r,R}$.

This convergence is smooth in the topology, and $C^{k+1,\alpha}$ in the metric by our L^∞ bounds on the k^{th} derivative of curvature. We get diffeomorphisms $\Phi_{i,r,R} : \Omega_{\infty,r,R} \to \Omega_{i,r,R}$ for large i such that the pullback metrics $\Phi^*_{i,r,R} g_i$ converge smoothly to g_∞. Adjusting r will change the limit manifold, but the limit manifolds naturally embed in one another. Put

$$\begin{aligned}\Omega_{\infty,R} &= \bigcup_{0<r} \Omega_{\infty,r,R} \\ \Omega_{\infty} &= \bigcup_{R<\infty} \Omega_{\infty,R}.\end{aligned}$$

The local upper bound on volume growth insures that Ω_∞ can be completed by adding discrete points, which constitute the singular set S, which has cardinality at most Λ/ϵ_0. The result is a complete manifold-with-singularities $M_\infty = \Omega_\infty \cup S$. It is possible that S is empty, or that some points of S might be smooth points of M.

In theorem 4.1 of [TV1], Tian-Viaclovsky show essentially that a complete manifold-with-singularities with Euclidean volume growth and $|\,\mathrm{Rm}\,| \cong o(r^{-2})$ at infinity is in fact ALE. Their method of proof is geometric and will hold in any dimension for manifolds-with-singularities, though it is stated for 4-dimensional smooth manifolds (see theorem 4.1 of [TV1] and the comment immediately afterwards). In our setting, the volume growth lower bound is implied by the Sobolev constant bound and quadratic curvature decay is ensured by condition (*iii*). An assumption on $b_1(M)$ is not necessary due to the improvements in [TV3]. Thus our limit manifold, if noncompact, will be ALE. $\square$

Next we examine the curvature singularities that arise in the limit, and, following the existing literature, sketch a proof that they are indeed C^∞ (possibly nonreduced) orbifold points. As is common in the literature, we say an orbifold possesses some structure if the structure exists at smooth points and, after lifting to the smooth orbifold cover of any point, it can be completed. For instance, an orbifold is called extremal Kähler if the lift of the metric to the orbifold cover of any point can be completed.

We also consider the *order* of the multifold points that arise in the limit. We define order as follows: if o is a multifold point with tangent cone T at o, the order of o is just the cardinality of the set of components of $T - \{o\}$. We will often use the terms orbifold and multifold interchangably. When it is important to distinguish, we shall call an orbifold *reduced* when each singular point has order 1.

Proposition 5.2 *Assume M is a Riemannian manifold-with-singularities and that M carries an extremal Kähler metric at every smooth point. Then the singularities are C^∞ Riemannian multifold points, and M is an extremal Kähler multifold. Further, the cardinality of any orbifold group Γ has a bound $|\Gamma| \le C(C_S)$, and the order of the multifold points are bounded by $C = C(C_S, \overline{v})$.*

Pf

This is a local proof; we need only consider neighborhoods of singularities.

Most of the work here is identical to that found elsewhere in the literature. Let o be a singularity. First choose a locally connected component N of $M - \{o\}$; by this it is meant that $(N \cap B(o, r)) - \{o\}$ is connected for all $r > 0$. We know that $|\operatorname{Rm}| = o(r^{-2})$ on $\overline{N}$, where r is the distance to o, and so the proof Lemma (5.13) of [BKN] yields that $\overline{N}$ has a unique tangent cone at o that is diffeomorphic to $\mathbb{R}^n/\Gamma$ where Γ is some isometric action on $\mathbb{R}$ whose only fixed point is $\{o\}$. Since the Sobolev inequality holds on $\overline{N}$ and hence $\overline{N}$ has local volume growth lower bounds, there is a bound on the cardinality $|\Gamma|$ of the orbifold group that depends only on n and C_S. Lastly, at any singularity point $\{o\}$, any small ball $B(o, r)$ must have at most a uniformly bounded number of components. This is because each component has local volume growth lower bounds, so many components together would give a very large local volume growth; this is impossible by assumption.

Now we examine the regularity of the metric on the orbifold cover of any component of a multifold point. Let $B = B(o, \epsilon)$ be a small ball around o diffeomorphic to T. Choose one component of $B - \{o\}$ and consider its orbifold cover (a neighborhood of the origin in $\mathbb{R}^n$). Lifting the metric to this neighborhood, we must analyze the regularity of the metric at a deleted point of $\mathbb{R}^n$.

With bounded curvature and dimension $n > 2$, elementary arguments show the metric is C^0. A less elementary argument suffices to construct $C^{1,1}$ coordinates; for instance the construction of [BKN] beginning on pg 342 shows this to be possible. We are able to cite this result in the higher dimensional case by Theorem 4.1, and in dimension 4 by Theorem 4.19. With $C^{1,1}$ coordinates, it is possible to construct harmonic coordinates, as in [DK].

In harmonic coordinates, we have the coupled system

$$\begin{aligned} \triangle(g_{ij}) &= \operatorname{Ric}_{ij} + Q(g, \partial g) \\ \triangle \operatorname{Ric} &= \operatorname{Rm} * \operatorname{Ric} + \nabla X \\ \triangle X &= \operatorname{Ric} * X. \end{aligned}$$

A bootstrapping argument is possible using the L^p theory. Since $\operatorname{Ric} \in L^p$ for some $p > 2$ and g and ∂g are bounded, the first equation gives $g_{ij} \in W^{2,p}$ for all p, so $\operatorname{Ric} \in W^{0,p}$. Then the third equation gives $(\nabla X)_{ij} \in W^{1,p}$ and so the second equation gives $\operatorname{Ric}_{ij} \in W^{2,p}$. Then the first equation gives $g_{ij} \in W^{4,p}$ and therefore by Sobolev imbedding $g_{ij} \in C^{3,\alpha}$, so $\operatorname{Rm} \in C^{1,\alpha}$. Now we turn to the Schauder theory. In harmonic coordinates the coefficients of $\triangle$ are $C^{3,\alpha}$, so the last equation gives $X_i \in C^{3,\alpha}$, so $(\nabla X)_{ij} \in C^{2,\alpha}$, so the middle equation gives $\operatorname{Ric}_{ij} \in C^{3,\alpha}$. Then with $\partial g \in C^{2,\alpha}$, the first equation gives $g_{ij} \in C^{4,\alpha}$, an improvement in regularity. Bootstrapping like this gives $g_{ij} \in C^{k,\alpha}$ for all k, so $g \in C^\infty$. All of this is standard elliptic theory; see for instance chapter 5 of [Evn] for Sobolev embedding, and chapters 6 and 9 of [GT] for the Schauder theory and L^p theory. This completes the proof that our curvature singularities are C^∞ Riemannian multifold points.

Finally we check the complex structure on the orbifold covers. Since the tensor J is harmonic (indeed, covariant constant) and of bounded norm, its lift will extend smoothly over the deleted point. The completed complex structure is clearly integrable, since the Nijenhuis tensor is smooth and is assumed to vanish everywhere except at the origin, and so it vanishes everywhere. Also, since $R_{,\bar{i}\bar{j}} = 0$ outside the singularity and R is C^∞, after completion $R_{,\bar{i}\bar{j}} = 0$ everywhere, so the metric on the orbifold cover is extremal Kähler. □

Proposition 5.3 (Limits are reduced orbifolds) *Suppose $(M_\alpha, g_\alpha, x_\alpha)_{\alpha\in A}$ is a family of n-dimensional extremal Kähler manifolds that satisfy the conditions (i)-(v) of this section, and which also have a local volume growth upper bound. Then a subsequence converges to a reduced extremal Kähler orbifold. If Γ is an orbifold group then $\Gamma \subset U(n)$, and there is a bound on its order, $|\Gamma| \le C(C_S, n)$. There is a bound the number of orbifold points, given by $C = C(n, \Lambda, C_S)$.*

<u>*Pf*</u>

Proposition (5.2) shows that any manifold-with-singularities constructed in the proof of proposition (5.1) is indeed a Riemannian multifold. We need only pass to a further subsequence to get a converging almost complex structure. The limiting complex structure is clearly integrable, since the Nijenhuis tensor will continue to vanish at all smooth points of the limit. C^1 convergence at smooth points implies also $d\omega = 0$ (where ω is the Kähler form), so the limiting multifold is Kähler at smooth points, and C^4 convergence guarantees that $R_{,\bar{i}\bar{j}} = 0$, so the multifold metric is extremal at smooth points. The orbifold group is a subgroup of $U(n)$ because its action on the cover preserves J.

Finally we establish that the limit is actually a *reduced* orbifold, meaning that $B(o,r) - \{o\}$ has only one component regardless of o or r. To this end we do a blowup analysis at a forming singularity in order to capture a two (or more) ended singularity model. Assume $\{o\}$ is a singularity that locally separates M_∞. Let $p_i \in M_i$ be a sequence of points with $p_i \to \{o\}$, and let $B(p_i, r_i)$ be balls with the following property: $\partial B(p_i, r_i)$ has one component, but whenever $r_i < \rho < i\, r_i$ then $\partial B(p_i, \rho)$ has more than one component (one must generally pass to a subsequence here).

Now rescale the manifolds M_i by setting $\bar{g}_i = r_i^{-2} g_i$, and take a limit. By Proposition 5.1 we know the limit is an ALE manifold-with-singularities, which we know are C^∞ orbifold points by Proposition 5.2. Since $B(o, 0.99) \subset M_\infty$ does not separate M_∞ but any ball $B(o,r)$ of radius $r > 1$ has more than one boundary component, we know the limit has more than one end. If the limit has a locally separating singularity, repeat the process until we arrive at a limiting object whose singularities do not locally separate.

We can use Theorem 4.1 of [LT1] to conclude that M has at most one non-parabolic end, and therefore at least one end is parabolic. However the result of Holopainen-Koskela or of Li-Tam (Theorem 1.4 of [HK], Theorem 1.9 of [LT2]) imply that none of our ends are parabolic. This contradiction establishes the proof. □

There has been a great deal of work relating the function-theoretic aspects of manifolds to their Riemannian or Kählerian geometry. We'd like to mention the nice survey article [Li] by Peter Li.

Finally we are able to complete the proof of Theorem 1.3 or 5.6 by removing condition (v) from the list at the beginning of this section. Following the proof of [TV2], we prove Theorem 5.5 for the case of extremal Kähler metrics. First we cite a volume comparison lemma; see for instance Proposition 20 of [Bor].

Lemma 5.4 (Orbifold volume comparison) *Assume M^n is a smooth Riemannian orbifold. Let $B = B(p,r) \subset M$ be any ball. If $\mathrm{Ric} \geq -(n-1)H$ in $B - S$, then $\mathrm{Vol}\, B(p,r) \leq \mathrm{Vol}_{-H} B(r)$.*

Define the maximal volume ratio $MV_t(M)$ of M^n at scale t to be

$$MV_t(M) = \sup_{x \in M,\, 0<r<t} r^{-n}\, \mathrm{Vol}\, B(x,r).$$

$MV_\infty(M)$ is of course an upper bound on the volume ratio of balls in M. We will also denote by $\mathrm{Vol}_c\, B(t)$ the volume of the ball of radius t in the space form of constant sectional curvature c.

Theorem 5.5 (Upper bound on volume growth) *Let $(M_\lambda, g_\lambda)_{\lambda \in A}$ be a family of compact, extremal Kähler manifolds. Assume $\mathrm{Vol}_{g_\lambda}(M_\lambda) \geq \nu$, $\mathrm{Diam}_{g_\lambda}(M_\lambda) \leq \delta$, $\| \mathrm{Rm} \|_{L^{\frac{n}{2}}} \leq \Lambda$, and Sobolev constants C_{M_λ} bounded above by $C_S < \infty$. Then there exists a bound on $MV_\infty(M_\lambda)$ depending on C_b, ν, δ, Λ, and C_S.*

Pf

Assume no such bound exists, so there is a sequence of such Riemannian manifolds $M_i = \{M, g_i\}$ with $MV_\infty(M_i) \to \infty$.

First, $\int_{B(x,2r)} |\mathrm{Rm}|^2 \leq \epsilon_0$ implies $|\mathrm{Rm}| \leq C\epsilon_0 r^{-2}$ in $B(x,r)$, so assuming (without loss of generality) that $r \leq \delta$, Bishop volume comparison gives $\mathrm{Vol}\, B(x,r) \leq r^n \delta^{-n}\, \mathrm{Vol}_{-C\epsilon_0} B(\delta) \triangleq Ar^n$.

Choose points $x_i \in M_i$ and radii r_i so that $\mathrm{Vol}\, B(x_i, r_i) = 2A(r_i)^4$, and r_i has the following minimality property: whenever $p \in M_i$ and $r \leq r_i$, we have $\mathrm{Vol}\, B(p,r) \leq 2Ar^4$. In other words, $MV_{r_i}(M_i) = 2A$. Note also that $\int_{B(x_i, 2r_i)} |\mathrm{Rm}|^2 \geq \epsilon_0$.

Set $x_i^{(1)} = x_i$, $r_i^{(1)} = r_i$, and $A^{(1)} = 2A = 2\,\mathrm{Vol}_{-C\epsilon_0} B(1)$. For an induction argument assume that k sequences of balls $\{B_i^{(1)}\}_{i=1}^{\infty}, \ldots, \{B_i^{(k)}\}_{i=1}^{\infty}$ have been chosen, where $B_i^{(j)} \triangleq B(x_i^{(j)}, r_i^{(j)})$ and $B_i^{(j)} \subset M_i$, and assume the balls satisfy the following assumptions:

- the balls in the j^{th} sequence, $B_i^{(j)}$, have volume ratio $(r_i^{(j)})^{-n} \cdot \mathrm{Vol}\, B_i^{(j)} \triangleq A^{(j)}$ fixed independent of i
- $A^{(j+1)} \geq 2A^{(j)}$
- each ball $B_i^{(j)}$ has the largest volume ratio among all balls in M_i of equal or smaller radius.
- for large i, $\int_{\bigcup_{j=1}^{k} B_i^{(j)}} |\mathrm{Rm}|^{\frac{n}{2}} \geq k\epsilon_0$,

For an induction argument we will show it is possible to extract a $(k+1)^{th}$ sequence with the same assumptions.

Choose one of the sequences $\{B_i^{(l)}\}_{i=1}^{\infty}$; we garner geometric information around the points $x_i^{(l)}$ by blowing up with $x_i^{(l)}$ as the basepoint. Scale each manifold M_i so that the i^{th} ball has radius 1, by setting $\tilde{g}_i = (r_i^{(l)})^{-2} g_i$. With the new metrics, we have an upper bound on the volume ratio for balls of radius ≤ 1, so after passing to a subsequence we get convergence to a limit multifold (M_∞, g_∞). We know the limit multifold is ALE and therefore has a global upper bound on volume growth, meaning $MV_\infty(M_\infty) \triangleq L^{(l)} < \infty$. Obviously $L^{(l)} \geq A^{(l)}$, so also $L^{(l)} \geq 2^l A$. We will denote the scaled radii $\tilde{r}_i^{(j)} = r_i^{(j)}/r_i^{(l)}$ and the scaled balls $\tilde{B}_i^{(j)} = \tilde{B}(x_i^{(j)}, \tilde{r}_i^{(j)})$. Of course $\tilde{B}^{(l)} = \tilde{B}(x_i^{(l)}, 1)$.

Return now to the unscaled manifolds. Choose a $(k+1)^{th}$ sequence of balls, $B_i^{(k+1)} = B(x_i^{(k+1)}, r_i^{(k+1)})$, with volume ratio $A^{(k+1)} = 2 \cdot 6^4 \cdot L^{(k)} (\geq 2A^{(k)})$, and so that $B_i^{(k+1)}$ has the largest volume ratio among all balls of equal or smaller radius. We will prove that for large i, $\int_{\bigcup_{j=1}^{k+1} B(x_i^{(j)}, 2r_i^{(j)})} |\mathrm{Rm}|^2 \geq (k+1)\epsilon_0$. This completes the induction argument and yields a contradiction with the $L^{\frac{n}{2}}$ curvature bound on the M_i.

We know that $r_i^{(k+1)} > r_i^{(l)}$, $l \leq k$. But we don't know whether $r_i^{(k+1)}/r_i^{(l)}$ is bounded.

Case I: If $r_i^{(k+1)}/r_i^{(l)}$ is bounded, $B(x_i^{(k+1)}, 2r_i^{(k+1)})$ is eventually disjoint from $B(x_i^{(l)}, 2r_i^{(l)})$

Assuming $r_i^{(k+1)}/r_i^{(l)} = \tilde{r}_i^{(k+1)} \leq N$,we can prove that for large i, $\tilde{B}(x_i^{(l)}, 2)$ and $\tilde{B}(x_i^{(k+1)}, 2\tilde{r}_i^{(k+1)})$ are disjoint. When i is large, regions in $(M, \tilde{g}_i)$ around

$x_i^{(l)}$ are very close to the limiting orbifold, so we have volume ratio bounds for balls near $x_i^{(l)}$: in particular, with i large we have $\mathrm{Vol}_{\tilde{g}_i}\,\tilde{B}(x_i^{(l)},\tilde{r}) < 2L^{(l)}\tilde{r}^4$ for arbitrary $\tilde{r} \leq 6N$.

Balls of bounded radius a bounded distance away from $x_i^{(l)}$ must have curvature ratio nearly bounded by $L^{(l)}$, the global volume ratio of the limit orbifold. Specifically, if $\tilde{B}(x_i^{(l)},2) \cap \tilde{B}(x_i^{(k+1)},2\tilde{r}_i^{(k+1)}) \neq \varnothing$, then $\tilde{B}(x_i^{(k+1)},2\tilde{r}_i^{(k+1)}) \subset \tilde{B}(x_i^{(l)},6\tilde{r}_i^{(k+1)})$ so $\mathrm{Vol}\,\tilde{B}(x_i^{(k+1)},\tilde{r}_i^{(k+1)}) \leq \mathrm{Vol}\,\tilde{B}(x_i^{(l)},6\tilde{r}_i^{(k+1)}) < 2L^{(l)}6^4(\tilde{r}_i^{(k+1)})^4$, a contradiction. Unscaling now, we have that $B(x_i^{(k+1)},2r_i^{(k+1)})$ is disjoint from $B(x_i^{(l)},2r_i^{(l)})$.

If this argument works for all $l \leq k$, we have $\int_{\bigcup_{l=1}^{k+1} B(x_i^{(l)},r_i^{(l)})} |\mathrm{Rm}|^2 \geq (k+1)\epsilon_0$ as desired. If not, for any l with $r_i^{(k+1)}/r_i^{(l)}$ unbounded, we move to the second case.

Case II: If $r_i^{(k+1)}/r_i^{(l)}$ is not bounded, $B(x_i^{(k+1)},r_i^{(k+1)})$ eventually has a region of high curvature that is disjoint from any of the other balls

Passing to a subsequence, we can assume $\tilde{r}_i^{(k+1)} \to \infty$. The idea is that if $\tilde{r}_i^{(k+1)}$ becomes unboundedly large, we might not have disjointness of $B(x_i^{(k+1)},2\tilde{r}_i^{(k+1)})$ from the smaller balls, but the smaller balls are nearly multifold points and therefore can only multiply the overall volume ratio by a controlled amount. Since the volume ratio is very large in $B_i^{(k+1)}$, volume comparison forces some region inside $B_i^{(k+1)}$ to have large $|\mathrm{Rm}|$ and therefore large $L^{\frac{n}{2}}$-norm of curvature disjoint from the other balls.

Assume $\int_{B(x_i^{(k+1)},2r_i^{(k+1)})-\bigcup_{j=1}^{l} B(x_i^{(j)},2r_i^{(j)})} |\mathrm{Rm}|^2 \leq \epsilon_0$ for all large i, for if not we are done. We know the unscaled radii $r_i^{(k+1)}$ are bounded as $i \to \infty$, so we can do a blowup limit by scaling $\tilde{g}_i = \left(r_i^{(k+1)}\right)^{-2} g_i$ and reach a limit orbifold $(\tilde{M}_\infty,\tilde{g}_\infty)$ with basepoint $x_\infty^{(k+1)}$. We can assume $\tilde{B}(x_\infty^{(k+1)},2)$ has k or fewer orbifold points, which correspond to the limits of the centers of the balls $\tilde{B}_i^{(j)}$. For if there are than $k+1$ multifold points, there are at least $k+1$ curvature concentration points in $B(x_i^{(k+1)},2r_i^{(k+1)})$, and we are done.

Letting S be the singular set in $\tilde{M}_\infty$, Fatou's lemma ensures we have $\int_{\tilde{B}(x_\infty^{(k+1)},2)-S} |\mathrm{Rm}|^2 \leq \epsilon_0$. The Moser iteration argument works despite the presence of orbifold points, and we have that $|\mathrm{Rm}| \leq C\epsilon_0$ on $\tilde{B}$, so orbifold volume comparison now guarantees that

$$\mathrm{Vol}\,\tilde{B} \;\leq\; \mathrm{Vol}_{-C\epsilon_0}B(1),$$

which violates the fact that we chose the $B_i^{(k+1)}$ to have volume ratio $\geq 2^{k+1}A =$

$2^{k+1}\operatorname{Vol}_{-C\epsilon_0} B(1)$. In the unscaled manifold we must have

$$\int_{B(x_i^{(k+1)},2r_i^{(k+1)})-\bigcup_{j=1}^k B(x_i^{(j)},2r_i^{(j)})} |\operatorname{Rm}|^2 \quad \geq \quad \epsilon_0,$$

so therefore $\int_{\bigcup_{j=1}^{k+1} B(x_i^{(j)},2r_i^{(j)})} |\operatorname{Rm}|^2 \geq (k+1)\epsilon_0$ □

Theorem 5.6 (Orbifold compactness) *Any family $\{M_\alpha, J_\alpha, g_\alpha\}_{\alpha\in A}$ of extremal manifolds satisfying conditions (i) - (iv) of the introduction contains a subsequence $\{M_i, J_i, g_i\}$ that converges in the Gromov-Hausdorff topology to a reduced compact extremal Kähler orbifold. Further, there is a bound $C_1 = C_1(\Lambda, C_S, n)$ on the number of singularities, and a bound $C_2 = C_2(C_S, n)$ on the order of any orbifold group.*

Pf

In light of Theorem 5.5, Proposition 5.3 now goes through without a separate assumption on local volume growth. □

In light of our results so far, an almost trivial consequence is the following gap theorem. Such a theorem is useful, for instance, in constructing bubble-trees. We state it here for convenience.

Corollary 5.7 (Gap theorem) *There exists an $\epsilon_0 = \epsilon_0(n, C_S)$ with the following property. Assuming (M, g, J) is an extremal Kähler orbifold (possibly nonreduced) and that*

$$\int_M |\operatorname{Rm}|^{\frac{n}{2}} \leq \epsilon_0,$$

then (M, g) is flat.

Pf

If the ϵ-regularity theorem can be shown to hold, namely that

$$\int_{B(o,r)} |\operatorname{Rm}|^{\frac{n}{2}} \leq \epsilon_0 \quad \Longrightarrow \quad |\operatorname{Rm}(o)| \leq Cr^{-2}\left(\int_{B(o,r)} |\operatorname{Rm}|^{\frac{n}{2}}\right)^{\frac{2}{n}},$$

then the result follows. If singularities are present, it is possible that the Moser iteration technique will fail, due to its reliance on integration by parts. However Theorems 4.1 and 4.20, combined with Sibner's lemma, Proposition 3.4, will ensure that residues will not crop up. Thus the Moser iteration goes through, and we get our result. □

6 Appendix: Local integral bounds for curvature at smooth points

6.1 Statement of the technical estimates

The following propositions hold when $\operatorname{supp}\phi$ consists of smooth points, and the real dimension is 4 or higher.

Lemma 6.1 *Assume $0 \leq \phi$ and $\int |\operatorname{Rm}|^{\frac{n}{2}}$ has been chosen small compared to C_S, k, and l. With $C = C(k, l, C_S)$ we have the two estimates*

$$\begin{aligned} \int \phi^l |\nabla T|^{k-2} |\nabla^2 T|^2 \quad &\leq \quad C \int \phi^{l-2} |\nabla \phi|^2 |\nabla T|^k \\ &\quad + C \int \phi^l |\nabla T|^{k-2} |\triangle T|^2 \\ &\quad + C \int \phi^l |\nabla T|^{k-2} |T|^2 |\operatorname{Rm}|^2 \end{aligned} \tag{59}$$

$$\begin{aligned} \int \phi^l |\nabla T|^{k-2} |\nabla^2 T|^2 \quad &\leq \quad C \int \phi^{l-2} |\nabla \phi|^2 |\nabla T|^k \\ &\quad + C \int \phi^l |\nabla T|^{k-2} |\triangle T|^2 \\ &\quad + C \int \phi^l |\nabla T|^{k-1} |T| |\nabla \operatorname{Ric}| \end{aligned} \tag{60}$$

Lemma 6.2 *Assume $0 \leq \phi$ and $\int |\operatorname{Rm}|^{\frac{n}{2}}$ has been chosen small compared to C_S, k, and l. With $C = C(k, l, C_S)$ we have the two estimates*

$$\begin{aligned} \left(\int \phi^{l\gamma} |\nabla T|^{k\gamma} \right)^{\frac{1}{\gamma}} \quad &\leq \quad C \int \phi^{l-2} |\nabla \phi|^2 |\nabla T|^k \\ &\quad + C \int \phi^l |\nabla T|^{k-2} |\triangle T|^2 \\ &\quad + C \int \phi^l |\nabla T|^{k-2} |T|^2 |\operatorname{Rm}|^2 \end{aligned} \tag{61}$$

$$\begin{aligned} \left(\int \phi^{l\gamma} |\nabla T|^{k\gamma} \right)^{\frac{1}{\gamma}} \quad &\leq \quad C \int \phi^{l-2} |\nabla \phi|^2 |\nabla T|^k \\ &\quad + C \int \phi^l |\nabla T|^{k-2} |\triangle T|^2 \\ &\quad + C \int \phi^l |\nabla T|^{k-1} |T| |\nabla \operatorname{Ric}| \end{aligned} \tag{62}$$

Lemma 6.3 *Assume* $0 \le \phi$, $|\nabla\phi| \le \frac{2}{r}$, *and* $\int |\operatorname{Rm}|^{\frac{n}{2}}$ *has been chosen small compared to* C_S, k, *and* l. *With* $C = C(k, l, C_S)$ *we have the two estimates*

$$\begin{aligned} \int \phi^l |\nabla T|^k \;\le\; & r^{-2} C \int \phi^{l-2} |\nabla T|^{k-2} |T|^2 \\ & + r^2 C \int \phi^{l+2} |\nabla T|^{k-2} |\triangle T|^2 \\ & + r^2 C \int \phi^{l+2} |\nabla T|^{k-2} |T|^2 |\operatorname{Rm}|^2 \end{aligned} \tag{63}$$

$$\begin{aligned} \int \phi^l |\nabla T|^k \;\le\; & r^{-2} C \int \phi^{l-2} |\nabla T|^{k-2} |T|^2 \\ & + r^2 C \int \phi^{l+2} |\nabla T|^{k-2} |\triangle T|^2 \\ & + r^2 C \int \phi^{l+2} |\nabla T|^{k-1} |T| |\nabla \operatorname{Ric}| \end{aligned} \tag{64}$$

6.2 Proof of the technical estimates

We achieve the estimates in a number of stages. Our spaghetti-like argument involves obtaining partial estimates for one quantity in order to estimate a second, and using the second to get a better estimate for the first, etc. The steps involved are standard, so exposition is kept to a minimum.

<u>*Initial estimate for* $\int \phi^l |\nabla T|^{k-2} |\nabla^2 T|^2$</u>

$$\begin{aligned} \int \phi^l |\nabla T|^{k-2} |\nabla^2 T|^2 \;=\; & -l \int \phi^{l-1} |\nabla T|^{k-2} \left\langle \nabla^2 T, \nabla\phi \otimes \nabla T \right\rangle \\ & - (k-2) \int \phi^l |\nabla T|^{k-2} |\nabla |\nabla T||^2 \\ & - \int \phi^l |\nabla T|^{k-2} \langle \triangle \nabla T, \nabla T \rangle \end{aligned}$$

$$\int \phi^l |\nabla T|^{k-2} |\nabla^2 T|^2 \;\le\; 2l^2 \int \phi^{l-2} |\nabla\phi|^2 |\nabla T|^k \,-\, 2 \int \phi^l |\nabla T|^{k-2} \langle \triangle \nabla T, \nabla T \rangle$$

<u>*Estimate for* $-\int \phi^l |\nabla T|^{k-2} \langle \triangle \nabla T, \nabla T \rangle$</u> Commutator formula:

$$\triangle \nabla T \;=\; \nabla \triangle T \,+\, \nabla(\operatorname{Rm} * T) \,+\, \operatorname{Rm} * \nabla T.$$

$$
\begin{aligned}
-\int \phi^l |\nabla T|^{k-2} \langle \triangle \nabla T, \nabla T \rangle \quad = \quad & -\int \phi^l |\nabla T|^{k-2} \langle \nabla \triangle T, \nabla T \rangle \\
& -\int \phi^l |\nabla T|^{k-2} \langle \nabla (\mathrm{Rm} * T), \nabla T \rangle \\
& -\int \phi^l |\nabla T|^{k-2} \langle \mathrm{Rm} * \nabla T, \nabla T \rangle .
\end{aligned}
$$

We estimate the three terms individually. First term:

$$
\begin{aligned}
& -\int \phi^l |\nabla T|^{k-2} \langle \nabla \triangle T, \nabla T \rangle \\
& = \ l \int \phi^{l-1} |\nabla T|^{k-2} \langle \nabla \phi \otimes \triangle T, \nabla T \rangle \\
& \quad + (k-2) \int \phi^l |\nabla T|^{k-3} \langle \nabla |\nabla T| \otimes \triangle T, \nabla T \rangle \\
& \quad + \int \phi^l |\nabla T|^{k-2} |\triangle T|^2
\end{aligned}
$$

$$
\begin{aligned}
& -\int \phi^l |\nabla T|^{k-2} \langle \nabla \triangle T, \nabla T \rangle \\
& \leq \ \frac{l^2}{2} \int \phi^{l-2} |\nabla \phi|^2 |\nabla T|^k \\
& \quad + \mu \int \phi^l |\nabla T|^{k-2} |\nabla^2 T|^2 \\
& \quad + \left(\frac{3}{2} + \frac{2(k-2)^2}{\mu} \right) \int \phi^l |\nabla T|^{k-2} |\triangle T|^2
\end{aligned}
$$

Second term:

$$
\begin{aligned}
& -\int \phi^l |\nabla T|^{k-2} \langle \nabla (\mathrm{Rm} * T), \nabla T \rangle \\
& = \ l \int \phi^{l-1} |\nabla T|^{k-2} \langle \nabla \phi \otimes (\mathrm{Rm} * T), \nabla T \rangle \\
& \quad + (k-2) \int \phi^l |\nabla T|^{k-3} \langle \nabla |\nabla T| \otimes (\mathrm{Rm} * T), \nabla T \rangle \\
& \quad + \int \phi^l |\nabla T|^{k-2} \langle \mathrm{Rm} * T, \triangle T \rangle
\end{aligned}
$$

$$
\begin{aligned}
&-\int \phi^l|\nabla T|^{k-2}\left\langle \nabla(\mathrm{Rm} * T), \nabla T\right\rangle \\
&\quad\leq\ \frac{l^2}{2}\int \phi^{l-2}|\nabla\phi|^2|\nabla T|^k \\
&\qquad + \mu\int \phi^l|\nabla T|^{k-2}|\nabla^2 T|^2 \\
&\qquad + \left(1+\frac{2(k-2)^2}{\mu}\right)\int \phi^l|\nabla T|^{k-2}|T|^2|\operatorname{Rm}|^2 \\
&\qquad + \frac{1}{2}\int \phi^l|\nabla T|^{k-2}|\triangle T|^2
\end{aligned}
$$

Third term:

$$
-\int \phi^l|\nabla T|^{k-2}\left\langle \mathrm{Rm} * \nabla T, \nabla T\right\rangle\ \leq\ \int \phi^l|\nabla T|^k|\operatorname{Rm}|
$$

Therefore

$$
\begin{aligned}
-\int \phi^l|\nabla T|^{k-2}\left\langle \triangle\nabla T, \nabla T\right\rangle \quad\leq\quad & l^2\int \phi^{l-2}|\nabla\phi|^2|\nabla T|^k \\
& + 2\mu\int \phi^l|\nabla T|^{k-2}|\nabla^2 T|^2 \\
& + \left(1+\frac{2(k-2)^2}{\mu}\right)\int \phi^l|\nabla T|^{k-2}|T|^2|\operatorname{Rm}|^2 \\
& + \left(2+\frac{2(k-2)^2}{\mu}\right)\int \phi^l|\nabla T|^{k-2}|\triangle T|^2 \\
& + \int \phi^l|\nabla T|^k|\operatorname{Rm}|.
\end{aligned}
$$

Alternative estimate for $-\int \phi^l|\nabla T|^{k-2}\left\langle \triangle\nabla T, \nabla T\right\rangle$

Commutator formula:

$$
\triangle\nabla T \quad=\quad \nabla\triangle T\ +\ \nabla\operatorname{Ric} * T\ +\ \mathrm{Rm} * \nabla T.
$$

$$
\begin{aligned}
-\int \phi^l|\nabla T|^{k-2}\left\langle \triangle\nabla T, \nabla T\right\rangle \quad=\quad & -\int \phi^l|\nabla T|^{k-2}\left\langle \nabla\triangle T, \nabla T\right\rangle \\
& -\int \phi^l|\nabla T|^{k-2}\left\langle \nabla\operatorname{Ric} * T, \nabla T\right\rangle \\
& -\int \phi^l|\nabla T|^{k-2}\left\langle \mathrm{Rm} * \nabla T, \nabla T\right\rangle.
\end{aligned}
$$

We deal with the terms individually. First term remains the same:

$$\begin{aligned}
&-\int \phi^l |\nabla T|^{k-2} \langle \nabla \triangle T, \nabla T\rangle \\
&\le \frac{l^2}{2}\int \phi^{l-2}|\nabla\phi|^2|\nabla T|^k \\
&\quad + \mu \int \phi^l |\nabla T|^{k-2}|\nabla^2 T|^2 \\
&\quad + \left(\frac{3}{2} + \frac{2(k-2)^2}{\mu}\right)\int \phi^l |\nabla T|^{k-2}|\triangle T|^2
\end{aligned}$$

Second term:

$$-\int \phi^l |\nabla T|^{k-2} \langle \nabla \operatorname{Ric} * T, \nabla T\rangle \quad \le \quad \int \phi^l |\nabla T|^{k-1}|T||\nabla \operatorname{Ric}|$$

The third term remains the same:

$$-\int \phi^l |\nabla T|^{k-2} \langle \operatorname{Rm} * \nabla T, \nabla T\rangle \ \le \ \int \phi^l |\nabla T|^k |\operatorname{Rm}|$$

Therefore

$$\begin{aligned}
-\int \phi^l |\nabla T|^{k-2} \langle \triangle \nabla T, \nabla T\rangle \quad \le \quad & \frac{l^2}{2}\int \phi^{l-2}|\nabla\phi|^2|\nabla T|^k \\
& + \mu \int \phi^l |\nabla T|^{k-2}|\nabla^2 T|^2 \\
& + \left(\frac{3}{2} + \frac{2(k-2)^2}{\mu}\right)\int \phi^l |\nabla T|^{k-2}|\triangle T|^2 \\
& + \int \phi^l |\nabla T|^k |\operatorname{Rm}| \\
& + \int \phi^l |\nabla T|^{k-1}|T||\nabla \operatorname{Ric}|
\end{aligned}$$

Two estimates for $\int \phi^l |\nabla T|^{k-2}|\nabla^2 T|^2$ First:

$$\begin{aligned}
\int \phi^l |\nabla T|^{k-2}|\nabla^2 T|^2 \quad \le \quad & 4l^2 \int \phi^{l-2}|\nabla\phi|^2|\nabla T|^k \\
& + \left(4 + 16(k-2)^2\right)\int \phi^l |\nabla T|^{k-2}|\triangle T|^2 \\
& + \left(2 + 16(k-2)^2\right)\int \phi^l |\nabla T|^{k-2}|T|^2|\operatorname{Rm}|^2 \\
& + 2\int \phi^l |\nabla T|^k |\operatorname{Rm}|
\end{aligned}$$

Second:

$$\begin{aligned}\int \phi^l |\nabla T|^{k-2}|\nabla^2 T|^2 \quad \leq \quad & 3l^2 \int \phi^{l-2}|\nabla\phi|^2|\nabla T|^k \\ & + \left(3 + 8(k-2)^2\right) \int \phi^l |\nabla T|^{k-2}|\triangle T|^2 \\ & + 2\int \phi^l |\nabla T|^{k-1}|T||\nabla \operatorname{Ric}| \\ & + 2\int \phi^l |\nabla T|^k |\operatorname{Rm}|\end{aligned}$$

Two Estimates for $\left(\int \phi^{l\gamma}|\nabla T|^{k\gamma}\right)^{\frac{1}{\gamma}}$

Sobolev Inequality:

$$\frac{1}{2C_S^2}\left(\int \phi^{l\gamma}|\nabla T|^{k\gamma}\right)^{\frac{1}{\gamma}} \quad \leq \quad l^2 \int \phi^{l-2}|\nabla\phi|^2|\nabla T|^k \,+\, k^2 \int \phi^l |\nabla T|^{k-2}|\nabla^2 T|^2$$

Assume

$$\left(\int |\operatorname{Rm}|^{\frac{n}{2}}\right)^{\frac{2}{n}} \leq \frac{1}{8k^2 C_S^2}.$$

First Estimate:

$$\begin{aligned}\left(\int \phi^{l\gamma}|\nabla T|^{k\gamma}\right)^{\frac{1}{\gamma}} \quad \leq \quad & 16k^2 l^2 C_S^2 \int \phi^{l-2}|\nabla\phi|^2|\nabla T|^k \\ & + 32k^4 C_S^2 \int \phi^l |\nabla T|^{k-2}|\triangle T|^2 \\ & + 32k^4 C_S^2 \int \phi^l |\nabla T|^{k-2}|T|^2|\operatorname{Rm}|^2 \\ & + 4k^2 C_S^2 \int \phi^l |\nabla T|^k |\operatorname{Rm}|\end{aligned}$$

$$\begin{aligned}\left(\int \phi^{l\gamma}|\nabla T|^{k\gamma}\right)^{\frac{1}{\gamma}} \quad \leq \quad & 32k^2 l^2 C_S^2 \int \phi^{l-2}|\nabla\phi|^2|\nabla T|^k \\ & + 64k^4 C_S^2 \int \phi^l |\nabla T|^{k-2}|\triangle T|^2 \\ & + 64k^4 C_S^2 \int \phi^l |\nabla T|^{k-2}|T|^2|\operatorname{Rm}|^2\end{aligned}$$

Second Estimate:

$$\left(\int \phi^{l\gamma}|\nabla T|^{k\gamma}\right)^{\frac{1}{\gamma}} \leq 8k^2l^2C_S^2\int \phi^{l-2}|\nabla\phi|^2|\nabla T|^k + 16k^4C_S^2\int \phi^l|\nabla T|^{k-2}|\triangle T|^2 + 4k^2C_S^2\int \phi^l|\nabla T|^{k-1}|T||\nabla \operatorname{Ric}| + 4k^2C_S^2\int \phi^l|\nabla T|^k|\operatorname{Rm}|$$

$$\left(\int \phi^{l\gamma}|\nabla T|^{k\gamma}\right)^{\frac{1}{\gamma}} \leq 16k^2l^2C_S^2\int \phi^{l-2}|\nabla\phi|^2|\nabla T|^k + 32k^4C_S^2\int \phi^l|\nabla T|^{k-2}|\triangle T|^2 + 8k^2C_S^2\int \phi^l|\nabla T|^{k-1}|T||\nabla \operatorname{Ric}|$$

Final estimate for $\int \phi^l|\nabla T|^{k-2}|\nabla^2 T|^2$

First:

$$\int \phi^l|\nabla T|^{k-2}|\nabla^2 T|^2 \leq C(k,l,C_S)\int \phi^{l-2}|\nabla\phi|^2|\nabla T|^k + C(k,l,C_S)\int \phi^l|\nabla T|^{k-2}|\triangle T|^2 + C(k,l,C_S)\int \phi^l|\nabla T|^{k-2}|T|^2|\operatorname{Rm}|^2$$

Second:

$$\int \phi^l|\nabla T|^{k-2}|\nabla^2 T|^2 \leq C(k,l,C_S)\int \phi^{l-2}|\nabla\phi|^2|\nabla T|^k + C(k,l,C_S)\int \phi^l|\nabla T|^{k-2}|\triangle T|^2 + C(k,l,C_S)\int \phi^l|\nabla T|^{k-1}|T||\nabla \operatorname{Ric}|$$

Initial estimate for $\int \phi^l|\nabla T|^k$

$$\begin{aligned}\int \phi^l |\nabla T|^k &= \int \phi^l |\nabla T|^{k-2} \langle \nabla T, \nabla T \rangle \\ &= -l \int \phi^{l-1} |\nabla T|^{k-2} \langle \nabla T, \nabla \phi \otimes T \rangle \\ &\quad -(k-2) \int \phi^l |\nabla T|^{k-3} \langle \nabla T, \nabla |\nabla T| \otimes T \rangle \\ &\quad - \int \phi^l |\nabla T|^{k-2} \langle \triangle T, T \rangle\end{aligned}$$

$$\begin{aligned}\frac{1}{2} \int \phi^l |\nabla T|^k &\leq \frac{l^2}{2} \int \phi^{l-2} |\nabla \phi|^2 |\nabla T|^{k-2} |T|^2 \\ &\quad + r^2 \frac{\mu}{2} \int \phi^{l+2} |\nabla T|^{k-2} |\nabla^2 T|^2 \\ &\quad + r^{-2} \frac{(k-2)^2}{2\mu} \int \phi^{l-2} |\nabla T|^{k-2} |T|^2 \\ &\quad + r^{-2} \frac{1}{2} \int \phi^{l-2} |\nabla T|^{k-2} |T|^2 \\ &\quad + r^2 \frac{1}{2} \int \phi^{l+2} |\nabla T|^{k-2} |\triangle T|^2\end{aligned}$$

$$\begin{aligned}\int \phi^l |\nabla T|^k &\leq \int \phi^{l-2} \left(l^2 |\nabla \phi|^2 + \mu^{-1} (k-2)^2 r^{-2} + r^{-2} \right) |\nabla T|^{k-2} |T|^2 \\ &\quad + r^2 \mu \int \phi^{l+2} |\nabla T|^{k-2} |\nabla^2 T|^2 \\ &\quad + r^2 \int \phi^{l+2} |\nabla T|^{k-2} |\triangle T|^2\end{aligned}$$

Two estimates for $\int \phi^l |\nabla T|^k$

In this section we assume $\phi \leq 1$ and $|\nabla \phi| \leq \frac{2}{r}$.

First estimate:

$$\begin{aligned}
\int \phi^l |\nabla T|^k \quad \leq \quad & \int \phi^{l-2} \left(l^2 |\nabla \phi|^2 + \mu^{-1}(k-2)^2 r^{-2} + r^{-2} \right) |\nabla T|^{k-2} |T|^2 \\
& + r^2 \mu 4 (l+2)^2 \int \phi^l |\nabla \phi|^2 |\nabla T|^k \\
& + r^2 \mu \left(4 + 16(k-2)^2 \right) \int \phi^{l+2} |\nabla T|^{k-2} |\triangle T|^2 \\
& + r^2 \mu \left(2 + 16(k-2)^2 \right) \int \phi^{l+2} |\nabla T|^{k-2} |T|^2 |\operatorname{Rm}|^2 \\
& + r^2 \mu 2 \int \phi^{l+2} |\nabla T|^k |\operatorname{Rm}| \\
& + r^2 \int \phi^{l+2} |\nabla T|^{k-2} |\triangle T|^2
\end{aligned}$$

$$\begin{aligned}
\int \phi^l |\nabla T|^k \quad \leq \quad & r^{-2} C(k,l) \int \phi^{l-2} |\nabla T|^{k-2} |T|^2 \\
& + r^2 C(k,l) \int \phi^{l+2} |\nabla T|^{k-2} |\triangle T|^2 \\
& + r^2 C(k,l) \int \phi^{l+2} |\nabla T|^{k-2} |T|^2 |\operatorname{Rm}|^2 \\
& + r^2 C(k,l) \left(\int \phi^{(l+2)\gamma} |\nabla T|^{k\gamma} \right)^{\frac{1}{\gamma}} \left(\int |\operatorname{Rm}|^{\frac{n}{2}} \right)^{\frac{2}{n}}
\end{aligned}$$

Continue:

$$\begin{aligned}
\int \phi^l |\nabla T|^k \quad \leq \quad & r^{-2} C(k,l) \int \phi^{l-2} |\nabla T|^{k-2} |T|^2 \\
& + r^2 C(k,l) \int \phi^{l+2} |\nabla T|^{k-2} |\triangle T|^2 \\
& + r^2 C(k,l) \int \phi^{l+2} |\nabla T|^{k-2} |T|^2 |\operatorname{Rm}|^2 \\
& + C(k,l,C_S) \left(\int \phi^l |\nabla T|^k \right) \left(\int |\operatorname{Rm}|^{\frac{n}{2}} \right)^{\frac{2}{n}} \\
& + r^2 C(k,l,C_S) \left(\int \phi^{l+2} |\nabla T|^{k-2} |\triangle T|^2 \right) \left(\int |\operatorname{Rm}|^{\frac{n}{2}} \right)^{\frac{2}{n}} \\
& + r^2 C(k,l,C_S) \left(\int \phi^{l+2} |\nabla T|^{k-2} |T|^2 |\operatorname{Rm}|^2 \right) \left(\int |\operatorname{Rm}|^{\frac{n}{2}} \right)^{\frac{2}{n}}
\end{aligned}$$

$$
\begin{aligned}
\int \phi^l |\nabla T|^k \quad \leq \quad & r^{-2} C(k,l,C_S) \int \phi^{l-2} |\nabla T|^{k-2} |T|^2 \\
& + r^2 C(k,l,C_S) \int \phi^{l+2} |\nabla T|^{k-2} |\triangle T|^2 \\
& + r^2 C(k,l,C_S) \int \phi^{l+2} |\nabla T|^{k-2} |T|^2 |\operatorname{Rm}|^2
\end{aligned}
$$

Second estimate:

$$
\begin{aligned}
\int \phi^l |\nabla T|^k \quad \leq \quad & \int \phi^{l-2} \left(l^2 |\nabla \phi|^2 + \mu^{-1}(k-2)^2 r^{-2} + r^{-2} \right) |\nabla T|^{k-2} |T|^2 \\
& + r^2 \mu 3(l+2)^2 \int \phi^l |\nabla \phi|^2 |\nabla T|^k \\
& + r^2 \mu \left(3 + 8(k-2)^2 \right) \int \phi^{l+2} |\nabla T|^{k-2} |\triangle T|^2 \\
& + r^2 \mu 2 \int \phi^{l+2} |\nabla T|^{k-1} |T| |\nabla \operatorname{Ric}| \\
& + r^2 \mu 2 \int \phi^{l+2} |\nabla T|^k |\operatorname{Rm}| \\
& + r^2 \int \phi^{l+2} |\nabla T|^{k-2} |\triangle T|^2
\end{aligned}
$$

$$
\begin{aligned}
\int \phi^l |\nabla T|^k \quad \leq \quad & r^{-2} C(k,l) \int \phi^{l-2} |\nabla T|^{k-2} |T|^2 \\
& + r^2 C(k,l) \int \phi^{l+2} |\nabla T|^{k-2} |\triangle T|^2 \\
& + r^2 C(k,l) \int \phi^{l+2} |\nabla T|^{k-1} |T| |\nabla \operatorname{Ric}| \\
& + r^2 C(k,l) \left(\int \phi^{(l+2)\gamma} |\nabla T|^{k\gamma} \right)^{\frac{1}{\gamma}} \left(\int |\operatorname{Rm}|^{\frac{n}{2}} \right)^{\frac{2}{n}}
\end{aligned}
$$

Continue:

$$\begin{aligned}\int \phi^l |\nabla T|^k \quad \leq \quad & r^{-2}C(k,l)\int \phi^{l-2}|\nabla T|^{k-2}|T|^2 \\ & + r^2 C(k,l)\int \phi^{l+2}|\nabla T|^{k-2}|\triangle T|^2 \\ & + r^2 C(k,l)\int \phi^{l+2}|\nabla T|^{k-1}|T||\nabla \operatorname{Ric}| \\ & + r^2 C(k,l,C_S)\left(\int \phi^l |\nabla\phi|^2|\nabla T|^k\right)\left(\int |\operatorname{Rm}|^{\frac{n}{2}}\right)^{\frac{2}{n}} \\ & + r^2 C(k,l,C_S)\left(\int \phi^{l+2}|\nabla T|^{k-2}|\triangle T|^2\right)\left(\int |\operatorname{Rm}|^{\frac{n}{2}}\right)^{\frac{2}{n}} \\ & + r^2 C(k,l,C_S)\left(\int \phi^{l+2}|\nabla T|^{k-1}|T||\nabla \operatorname{Ric}|\right)\left(\int |\operatorname{Rm}|^{\frac{n}{2}}\right)^{\frac{2}{n}}\end{aligned}$$

$$\begin{aligned}\int \phi^l |\nabla T|^k \quad \leq \quad & r^{-2}C(k,l,C_S)\int \phi^{l-2}|\nabla T|^{k-2}|T|^2 \\ & + r^2 C(k,l,C_S)\int \phi^{l+2}|\nabla T|^{k-2}|\triangle T|^2 \\ & + r^2 C(k,l,C_S)\int \phi^{l+2}|\nabla T|^{k-1}|T||\nabla \operatorname{Ric}|\end{aligned}$$

6.3 The induction argument in the smooth case

In this section we assume the following:

Hypothesis 6.4 *Assume $a \geq \frac{n}{2}$, $n \geq 4$. There exist $\epsilon_0 = \epsilon_0(C_S, p, a, n)$ and $C = C(C_S, p, a, n)$ so that if $\int_{B(o,r)} |\operatorname{Rm}|^{\frac{n}{2}} \leq \epsilon_0$, then*

$$\left(\int_{B(o,r/2)} |\nabla^{p-1}\operatorname{Rm}|^a\right)^{\frac{1}{a}} \leq Cr^{-1-p+\frac{n}{a}}\left(\int_{B(o,r)} |\operatorname{Rm}|^{\frac{n}{2}}\right)^{\frac{2}{n}} \tag{65}$$

$$\left(\int_{B(o,r/2)} |\nabla^{p}\operatorname{Ric}|^a\right)^{\frac{1}{a}} \leq Cr^{-2-p+\frac{n}{a}}\left(\int_{B(o,r)} |\operatorname{Ric}|^{\frac{n}{2}}\right)^{\frac{2}{n}} \tag{66}$$

$$\left(\int_{B(o,r/2)} |\nabla^{p} X|^a\right)^{\frac{1}{a}} \leq Cr^{-3-p+\frac{n}{a}}\left(\int_{B(o,r)} |R|^{\frac{n}{2}}\right)^{\frac{2}{n}}, \tag{67}$$

and prove:

Theorem 6.5 *Assume Hypothesis 6.4. There exist* $\epsilon_0 = \epsilon_0(C_S, p, a, n)$ *and* $C = C(C_S, p, a, n)$ *so that if* $\int_{B(o,r)} |\operatorname{Rm}|^{\frac{n}{2}} \leq \epsilon_0$*, then*

$$\left(\int_{B(o,r/2)} |\nabla^p \operatorname{Rm}|^a\right)^{\frac{1}{a}} \leq Cr^{-2-p+\frac{n}{a}} \left(\int_{B(o,r)} |\operatorname{Rm}|^{\frac{n}{2}}\right)^{\frac{2}{n}} \tag{68}$$

$$\left(\int_{B(o,r/2)} |\nabla^{p+1} \operatorname{Ric}|^a\right)^{\frac{1}{a}} \leq Cr^{-3-p+\frac{n}{a}} \left(\int_{B(o,r)} |\operatorname{Ric}|^{\frac{n}{2}}\right)^{\frac{2}{n}} \tag{69}$$

$$\left(\int_{B(o,r/2)} |\nabla^{p+1} X|^a\right)^{\frac{1}{a}} \leq Cr^{-4-p+\frac{n}{a}} \left(\int_{B(o,r)} |R|^{\frac{n}{2}}\right)^{\frac{2}{n}}. \tag{70}$$

Now Propositions 4.2, 4.3, 4.4, and 4.5 together show that Hypothesis 6.4 holds in the case $p = 0$, so the conclusion is true for all $p \in \mathbb{N}$.

Pf of Theorem 6.5

First we use the commutator formula (30) to get three estimates:

$$|\triangle \nabla^{p-1} \operatorname{Rm}|^2 \leq C \sum_{i=0}^{p-1} |\nabla^i \operatorname{Rm}|^2 |\nabla^{p-1-i} \operatorname{Rm}|^2 + C|\nabla^{p+1} \operatorname{Ric}|^2 \tag{71}$$

$$|\triangle \nabla^p \operatorname{Ric}|^2 \leq C \sum_{i=0}^{p} |\nabla^i \operatorname{Rm}|^2 |\nabla^{p-i} \operatorname{Ric}|^2 + C|\nabla^{p+1} X|^2 \tag{72}$$

$$|\triangle \nabla^p X|^2 \leq C \sum_{i=0}^{p-1} |\nabla^i \operatorname{Rm}|^2 |\nabla^{p-i} X|^2 + C \sum_{i=0}^{p} |\nabla^i \operatorname{Ric}|^2 |\nabla^{p-i} X|^2. \tag{73}$$

Notice that $\triangle \nabla^p X$ involves at most the $(p-1)^{th}$ derivative of Rm.

Step I: Estimating $|\nabla^{p+1} X|$ *norms*

We deal with $\nabla^{p+1}X$ first. We can use Technical Lemma 6.3 to get

$$\begin{aligned}
\int \phi^k |\nabla^{p+1}X|^k \leq\ & Cr^{-k}\int \phi^k |\nabla^p X|^k \\
& + Cr^2 \left(\int \phi^{k\gamma}|\nabla^p X|^{k\gamma}\right)^{\frac{1}{\gamma}} \left(\int_{\mathrm{supp}(\phi)} |\nabla \operatorname{Ric}|^{\frac{n}{2}}\right)^{k\frac{2}{n}} \\
& + Cr^2 \int \phi^k |\nabla^{p+1}X|^{k-2}|\triangle \nabla^p X|^2 \\
\leq\ & Cr^{-k}\int \phi^k |\nabla^p X|^k \\
& + Cr^2 \left(\int \phi^{k\gamma}|\nabla^p X|^{k\gamma}\right)^{\frac{1}{\gamma}} \left(\int_{\mathrm{supp}(\phi)} |\nabla \operatorname{Ric}|^{\frac{n}{2}}\right)^{k\frac{2}{n}} \\
& + Cr^2 \sum_{i=0}^{p-1} \int \phi^k |\nabla^{p+1}X|^{k-2}|\nabla^i \operatorname{Rm}|^2 |\nabla^{p-i}X|^2 \\
& + Cr^2 \sum_{i=0}^{p} \int \phi^k |\nabla^{p+1}X|^{k-2}|\nabla^i \operatorname{Ric}|^2 |\nabla^{p-i}X|^2.
\end{aligned}$$

The induction hypothesis yields estimates for each of the $\nabla^i \operatorname{Rm}$, $\nabla^i \operatorname{Ric}$, and $\nabla^{p-i}X$ integral terms. Then using Hölder's inequality and collecting terms will give us

$$\int \phi^k |\nabla^{p+1}X|^k \leq Cr^{-k}\int \phi^k |\nabla^p X|^k.$$

Using the induction hypothesis again yields

$$\left(\int_{B(o,r/2)} |\nabla^{p+1}X|^k\right)^{\frac{1}{k}} \leq Cr^{-4-p+\frac{n}{k}} \left(\int_{B(o,r)} |R|^{\frac{n}{2}}\right)^{\frac{2}{n}}. \tag{74}$$

Step II: Estimating $|\nabla^{p+1} \operatorname{Ric}|$ norms

Now it is necessary to bound $\int |\nabla^{p+1} \operatorname{Ric}|^k$. Using 6.3 again, we get

$$\begin{aligned}
\int \phi^k |\nabla^{p+1}\operatorname{Ric}|^k \leq\ & Cr^{-2}\int \phi^{k-2} |\nabla^{p+1}\operatorname{Ric}|^{k-2}|\nabla^p \operatorname{Ric}|^2 \\
& + Cr^2 \int \phi^{k+2}|\nabla^{p+1}\operatorname{Ric}|^{k-2}|\nabla^p \operatorname{Ric}||\nabla \operatorname{Ric}| \\
& + Cr^2 \int \phi^{k+2}|\nabla^{p+1}\operatorname{Ric}|^{k-2}|\triangle \nabla^p \operatorname{Ric}|^2,
\end{aligned}$$

which, with the induction hypothesis and Hölder's inequality, becomes

$$\begin{aligned}\int \phi^k |\nabla^{p+1} \operatorname{Ric}|^k &\leq Cr^{-k} \int |\nabla^p \operatorname{Ric}|^k \\ &\quad + Cr^2 \int \phi^{k+2} |\nabla^{p+1} \operatorname{Ric}|^{k-2} |\triangle \nabla^p \operatorname{Ric}|^2.\end{aligned}$$

Use the formula for $\triangle \nabla^p \operatorname{Ric}$. Noting that all of the terms appearing in $\triangle \nabla^p \operatorname{Ric}$ are estimable except the $|\nabla^p \operatorname{Rm}|$ term, we can use Hölder's inequality to actually get

$$\begin{aligned}\int \phi^k |\nabla^{p+1} \operatorname{Ric}|^k &\leq Cr^{-k} \int_{\operatorname{supp} \phi} |\nabla^p \operatorname{Ric}|^k \\ &\quad + Cr^2 \int \phi^{k+2} |\nabla^{p+1} \operatorname{Ric}|^{k-2} |\nabla^p \operatorname{Rm}|^2 |\operatorname{Ric}|.\end{aligned} \tag{75}$$

We work with the final term:

$$\begin{aligned}&Cr^2 \int \phi^{k+2} |\nabla^{p+1} \operatorname{Ric}|^{k-2} |\nabla^p \operatorname{Rm}|^2 |\operatorname{Ric}| \\ &\leq Cr^2 \left(\int \phi^{(k+2)\gamma} |\nabla^{p+1} \operatorname{Ric}|^{k\gamma} \right)^{\frac{k-2}{k}\frac{n-2}{n}} \left(\int \phi^k |\nabla^p \operatorname{Rm}|^k \right)^{\frac{2}{k}} \left(\int \phi^{\frac{2n}{k-2}} |\operatorname{Ric}|^{\frac{k}{k-2}\frac{n}{2}} \right)^{\frac{k-2}{k}\frac{2}{n}}\end{aligned}$$

Now we must work with the $\left(\int \phi^{(k+2)\gamma} |\nabla^{p+1} \operatorname{Ric}|^{k\gamma}\right)^{\frac{k-2}{k}\frac{n-2}{n}}$ factor. Using Technical Lemma 6.2 we get

$$\begin{aligned}\left(\int \phi^{(k+2)\gamma} |\nabla^{p+1} \operatorname{Ric}|^{k\gamma} \right)^{\frac{n-2}{n}} &\leq C \int \phi^k |\nabla \phi|^2 |\nabla^{p+1} \operatorname{Ric}|^k \\ &\quad + C \int \phi^k |\nabla^{p+1} \operatorname{Ric}|^{k-2} |\triangle \nabla^p \operatorname{Ric}|^2 \\ &\quad + C \int \phi^k |\nabla^{p+1} \operatorname{Ric}|^{k-1} |\nabla^p \operatorname{Ric}| |\nabla \operatorname{Ric}|.\end{aligned}$$

The integral norms of all quantities except $|\nabla^{p+1} \operatorname{Ric}|$ are estimable, and we get

$$\left(\int \phi^{(k+2)\gamma} |\nabla^{p+1} \operatorname{Ric}|^{k\gamma} \right)^{\frac{n-2}{n}} \leq Cr^{-2} \int \phi^k |\nabla \phi|^2 |\nabla^{p+1} \operatorname{Ric}|^k.$$

Putting this back into (75) and using the induction hypothesis gives

$$\begin{aligned}\int \phi^k |\nabla^{p+1} \operatorname{Ric}|^k &\leq Cr^{-k} \int_{\operatorname{supp} \phi} |\nabla^p \operatorname{Ric}|^k \\ &\quad + Cr^{-k} \left(\int_{\operatorname{supp} \phi} |\operatorname{Ric}|^{\frac{n}{2}} \right)^{\frac{2}{n}} \left(\int \phi^k |\nabla^p \operatorname{Rm}|^k \right).\end{aligned} \tag{76}$$

Step III: Estimating $|\nabla^p \operatorname{Rm}|$ norms
Using Technical Lemma 6.3 we get

$$\begin{aligned}\int \phi^k|\nabla^p \operatorname{Rm}|^k &\leq Cr^{-2}\int \phi^{k-2}|\nabla^p \operatorname{Rm}|^{k-2}|\nabla^{p-1}\operatorname{Rm}|^2 \\ &+Cr^2\int \phi^{k+2}|\nabla^p \operatorname{Rm}|^{k-2}|\nabla^{p-1}\operatorname{Rm}|^2|\operatorname{Rm}|^2 \\ &+Cr^2\int \phi^{k+2}|\nabla^p \operatorname{Rm}|^{k-2}|\triangle\nabla^{p-1}\operatorname{Rm}|^2.\end{aligned}$$

Applying Hölder's inequality and the induction hypothesis, we get

$$\int \phi^k|\nabla^p \operatorname{Rm}|^k \leq Cr^{-k}\int |\nabla^{p-1}\operatorname{Rm}|^k + Cr^k\int \phi^{k+2}|\triangle\nabla^{p-1}\operatorname{Rm}|^k.$$

In fact the integral norms of all quantities appearing in $\triangle\nabla^{p-1}\operatorname{Rm}$ are estimable by the induction hypothesis. We are left with only

$$\int \phi^k|\nabla^p \operatorname{Rm}|^k \leq Cr^{-k}\int |\nabla^{p-1}\operatorname{Rm}|^k + Cr^k\int \phi^{k+2}|\nabla^{p+1}\operatorname{Ric}|^k,$$

which we can estimate with (76) to get

$$\int \phi^k|\nabla^p \operatorname{Rm}|^k \leq Cr^{-k}\int |\nabla^{p-1}\operatorname{Rm}|^k.$$

Finally the induction hypothesis gives

$$\int_{B(o,r/2)} |\nabla^p \operatorname{Rm}|^k \leq Cr^{-pk}\int_{B(o,r)} |\operatorname{Rm}|^k.$$

Finally also equation (75) and the result of Step II give the final estimate for $\int |\nabla^{p+1}\operatorname{Ric}|^k$. □

References

[And1] M. Anderson. *Ricci curvature bounds and Einstein metrics on Compact manifolds*. **Journal of the AMS** Vol. 2 (1989) No. 3, 455–490

[And2] M. Anderson. *Orbifold Compactness for spaces of Riemannian metrics and applications*. **Math. Ann.** Vol. 331 (2005) No. 4, 739-778

[And3] M. Anderson. *Canonical metrics on 3-manifolds and 4-manifolds*. **Math. Ann.** Vol 10 (2006) No. 1, 127-164

[ACGT] V. Apostolov, D. Calderbank, P. Gauduchon, C. Tonnesen-Friedman. *Hamiltonian 2-forms in Kähler geometry, III, extremal metrics and stability* **math.DG/0511118**

[AP] C. Arezzo, F. Pacard, *Blowing up and desingularizing Kähler orbifolds with constant scalar curvature* **math.DG/0504115**

[APS] C. Arezzo, F. Pacard, M. Singer. *Extremal metrics on blow ups* **math.DG/0511118**

[Aub] T. Aubin. *Equations du type de Monge-Ampére sur les variétés kählériennes compactes*, C.R. Acad. Sci. Paris, Vol 283 (1976) 119-121

[BKN] S. Bando, A. Kasue, H. Nakajima. *On a construction of coordinates at infinity with fast curvature decay and maximal volume growth*. **Invent. Math.** Vol. 97 (1989) 313–349

[Bor] Borzellino, J. E. *Orbifolds of maximal diameter*. **Indiana U. of Math. J.** Vol. 42 (1993), 37–53

[Bra] T. Branson, *Kato constants in Riemannian Geometry*. **Mathematical Research Letters** Vol. 7 (2000) No. 2, 245–262

[CGH] D. Calderbank, P. Gauduchon, M. Herzlich. *Refined Kato inequalities and conformal weights in Riemannian Geometry*. ArXiv preprint: math.DG/9909116

[CS] H.D. Cao, N. Sesum. *The compactness result for K ähler Ricci Solitons*. ArXiv preprint: math.DG/0504526

[Cal1] E. Calabi. *Extremal Kähler metrics*. **Seminar on Differential Geometry**, Vol 16 of 102. Ann. of Math. Studies, University Press, 1982. 259–290

[Cal2] E. Calabi. *Extremal Kähler metrics, II*. **Differential geometry and Complex Analysis**, Springer, 1985. 96–114

[CQY] S.A. Chang, J. Qing, and P. Yang. *On a conformal gap and finiteness theorem for a class of four manifolds*, ArXiv preprint: math.DG/0508621

[Che] J. Cheeger. *Finiteness Theorems for Riemannian Manifolds*. **Ann. of Math.** Vol. 92 (1970) No. 1, 61–74

[CCT] J. Cheeger, T.H. Colding, G. Tian. *On the singularities of spaces with bounded Ricci curvature*. **GAFA** Vol. 12 (2002) 873–914

[CT] J. Cheeger and G. Tian. *Curvature and injectivity radius estimates for Einstein 4-manifolds*. **Proceedings of the AMS** Vol 19, Iss 2 (2006) 487–525

[Chn1] X.X. Chen. *Space of Kähler metrics*. **J. Differenital Geom.** Vol. 56 (2000) 189–234

[Chn2] Xiuxiong Chen. *Space of Kahler metrics III - On the lower bound of the Calabi energy and geodesic distance*. ArXiv preprint: math.DG/0606228

[ChnT1] X.X. Chen, G. Tian. *Partial regularity for homogeneous complex Monge-Ampere equations.* (Engligh, French summary) **C. R. Math. Acad. Sci. Paris** Vol. 340 (2005) No. 5, 337–340

[ChnWeb2] X. X. Chen and B. Weber. *Convergence of Extremal Kähler metrics in Tian's cone*, in preparation.

[Don1] S.K. Donaldson. *Scalar curvature and stability of toric varieties.* **J. Diff. Geom.** Vol. 62 (2002) No. 2, 289–349

[Don2] S.K. Donaldson. *Interior estiamtes for solutions of Abreu's equation.* **Collect. Math.** Vol. 56 (2005) No. 2, 103–142

[Don3] S.K. Donaldson. *Extremal metrics on toric surfaces, I.* **math.DG/0612120**

[DK] DeTurck, Kazdan. *Some regularity theorems in Riemannian geometry.* Annales Scientifiques de l'...cole Normale Suprieure Sr. 4, Vol. 14 (1981) No. 3, 249–260

[Fine] Fine, Joel. *Constant scalar curvature Kähler metrics on fibered complex surfaces.* **J. Diff. Geom.** 68 (2004) No. 3, 397–432

[Fut] A. Futaki. *An obstruction to the existence of Einstein Kähler metrics.* **Invent. Math.** Vol. 73 (1983) No. 3, 437–443

[FM] A. Futaki and T. Mabuchi. *Bilinear forms and extremal Kähler vector fields associated with Kähler classes.* **Math. Ann.** Vol. 301 (1995) No. 2, 199–210

[GT] Gilbarg, Trudinger. Elliptic Partial Differential Equations of Second Order, 2nd Edition. *Grundlehren der mathematischen Wissenschaften*, 224. **Springer-Verlag** (1983)

[Gro] M. Gromov. *Structures métriques pour les variétés riemanniennes* eds. J. Lafontaine and P. Pansu, CEDIC, Paris, 1981.

[Evn] Evans, L. Partial Differential Equations. *Graduate Studies in Mathematics*, Vol 19. **AMS** (1998)

[HK] I. Holopainen, P. Koskela. *Volume growth and parabolicity.* **Proc. AMS** Vol. 129, No. 11, 3425-3435

[LS1] C. LeBrun and S.R. Simanca. *Extremal Kähler metrics and deformation theory.* **G.A.F.A.** Vol. 4 (1994) No. 3, 298–336

[LS2] C. LeBrun and S.R. Simanca. *On Kähler surfaces of positive scalar curvature.* **J. Geom. Anal.** Vol. 5 (1995) No. 1, 115–127

[Lev] M. Levine. *A Remark on extremal Kähler metrics.* **J. Differential Geom**. Vol. 21 (1985) No. 1, 73–77

[Li] P. Li. *Curvature and function theory on Riemannian manifolds.* **Surveys in Differential Geometry: Papers dedicated to Atiyah, Bott, Hirzebruch, and Singer**, Vol VII, International Press (2000) 375-432

[LT1] P. Li, L. Tam. *Harmonic functions and the structure of complete manifolds.* **J. Diff. Geom.** Vol. 35 (1992) 359-383

[LT2] P. Li, L. Tam. *Green's functions, harmonic functions, and volume comparison.* **J. Diff. Geom.** Vol. 41 (1995) 277-318

[Mab] T. Mabuchi. *Uniqueness of extremal Kähler metrics for an integral Kähler class.* (English summary) **Internat. J. Math.** Vol. 15 (2004) No. 6, 531–546

[Sib] L. Sibner. *Isolated point singularity problem.* **Mathematische Annalen** Vol. 271 (1985) 125–131

[Tia1] G. Tian. *On Calabi's conjecture for complex surfaces with positive first Chern class.* **Invent. Math.** Vol. 101 (1990) No. 1, 101–172

[Tia2] G. Tian. *Kähler-Einstein metrics with positive scalar curvature* **Invent. Math.** Vol. 130 (1997) No. 1 1–37

[Tia3] G. Tian. *moduli space of extremal Kähler metrics*, unpublished note, SUNY Stony Brook, 1992

[TV1] G. Tian, J. Viaclovsky. *Bach-flat asymptotically locally Euclidean metrics.* **Invent. Math.** Vol. 160, No. 2 (2005) 357–415

[TV2] G. Tian, J Viaclovsky. *Moduli spaces of critical Riemannian metrics in dimension four.* **Advances in Math.** Vol. 196 (2005) No. 2, 346–372

[TV3] G. Tian, J Viaclovsky. *Volume growth, curvature decay, and critical metrics.* ArXiv preprint: math.DG/0612491

[Yau] S.T. Yau. *On the Ricci curvature of a compact Kahler manifold and the complex Monge-Ampere equation I.* **Comm. Pure Appl. Math** (1978)

[Uhl] K. Uhlenbeck. *Removable singularities in Yang- Mills fields.* bf Comm. Math. Phys. Vol. 83 (1982) 11–29

[Web1] B. Weber. *Compactification of the Moduli Space of Extremal Kähler Metrics.* (Ph.D. thesis)

[Web2] B. Weber. *A note on Kähler-Ricci solitons.* (*in preparation.*)

[Zhu] B. Zhou and X. Zhu. *Relative K-stability and modified K-energy on toric manifolds*, ArXiv preprint: math.DG/0603237