The Sphere

The metric

Using the standard Euclidean metric on $\mathbb{R}^4$

$$\begin{aligned} g_E &= dx^1 \otimes dx^1 + dx^2 \otimes dx^2 + dx^3 \otimes dx^3 + dx^4 \otimes dx^4 \\ &= \delta_{ij} dx^i \otimes dx^j. \end{aligned} \tag{1}$$

Becuase $g_{E,ij} = \delta_{ij}$ we sometimes abbreviate $g_E = \delta$. Using the radial variable $r = \sqrt{(x^1)^2 + (x^2)^2 + (x^3)^2 + (x^4)^2}$ we define the conformally related metric

$$g = \frac{4}{(1+r^2)^2}\, g_E. \tag{2}$$

From our conformal change formulae for $\hat{g} = u^2 g$ with $u = 2/(1+r^2)$, we obtain auxilliary tensor and its trace

$$K_{ij} = \frac{2}{(1+r^2)^2}\delta_{ij}, \quad Tr\, K = \frac{8}{(1+r^2)^2} \tag{3}$$

and because $\mathrm{Rm}_E \equiv 0$, the curvature quantities are easy to compute using our conformal change formulas:

$$\begin{aligned} \mathrm{Rm} &= 4(1+r^2)^{-2}\,(K \owedge g_E) = 8(1+r^2)^{-4}\,\delta \owedge \delta = \frac{1}{2} g \owedge g \\ \mathrm{Ric} &= 12(1+r^2)^{-2}\,\delta = 3g \\ R &= 12. \end{aligned} \tag{4}$$

This is a metric of constant sectional curvature +1.

Coordinate tranformations

Strictly speaking, the metric (2) exists only on $\mathbb{R}^4 = \mathbb{S}^4 \setminus \{\infty\}$. To claim it really represents a metric on the sphere, we must prove that it extends smoothly across the point at infinity.

To obtain a coordinate chart that contains ∞, we define coordinate transitions

$$y^i \;=\; x^i/r^2. \tag{5}$$

If we define $\rho = \sqrt{(y^1)^2 + (y^2)^2 + (y^3)^2 + (y^4)^2}$, then $\rho = 1/r$. One easily computes $\delta_{ij} dx^i \otimes dx^j = \delta_{ij}\rho^{-4} dy^i \otimes dy^j$, and therefore

$$\frac{4}{(1+r^2)^2}\delta_{ij}dx^i \otimes dx^j \;=\; \frac{4}{(1+\rho^2)^2}\delta_{ij}dy^i \otimes dy^j. \tag{6}$$

The metric expressed in y-coordinates extends smoothly across the point at infinity (which is just $(0,0,0,0)$ in the y-system). We conclude that this is indeed a smooth metric on $\mathbb{S}^2$.

Exercises

1. Given constants α, β, γ, show that the metric

$$g = \frac{4\alpha^2}{(\beta^2 + \gamma^2 r^2)^2} g_E \tag{7}$$

gives a metric constant curvature $+\frac{\alpha}{\beta\gamma}$ on the sphere.

2. Compute the volume of the 4-sphere.

(Updated Oct 2018)