Hyperbolic 4-Space

The Poincare Ball

Using the standard Euclidean metric on $\mathbb{R}^4$

$$\begin{aligned} g_E &= dx^1 \otimes dx^1 + dx^2 \otimes dx^2 + dx^3 \otimes dx^3 + dx^4 \otimes dx^4 \\ &= \delta_{ij} dx^i \otimes dx^j. \end{aligned} \tag{1}$$

Because $g_{E,ij} = \delta_{ij}$ we sometimes abbreviate $g_E = \delta$. Using the radial variable $r = \sqrt{(x^1)^2 + (x^2)^2 + (x^3)^2 + (x^4)^2}$ we define the conformally related metric

$$g = \frac{4}{(1-r^2)^2} g_E. \tag{2}$$

From our conformal change formulas for $\hat{g} = u^2 g$ with $u = 2/(1-r^2)$, we obtain auxiliary tensor and its trace

$$K_{ij} = -\frac{2}{(1-r^2)^2}\delta_{ij}, \quad Tr\, K = -\frac{8}{(1-r^2)^2} \tag{3}$$

and because $\mathrm{Rm}_E \equiv 0$, the curvature quantities are easy to compute using our conformal change formulas:

$$\begin{aligned} \mathrm{Rm} &= 4(1-r^2)^{-2} (K \owedge g_E) = -8(1-r^2)^{-4}\, \delta \owedge \delta = -\frac{1}{2} g \owedge g \\ \mathrm{Ric} &= -12(1-r^2)^{-2}\, \delta = -3g \\ R &= -12. \end{aligned} \tag{4}$$

This is a metric of constant sectional curvature -1.

The metric (2) exists only on the ball $\{r < 1\} \subset \mathbb{R}^4$. The most interesting remaining question is whether the metric is complete; this can be verified easily by estimating the lengths of arbitrary smooth paths to the boundary.

The Upper Half Space

Again we start with the Euclidean metric $g_E = \delta$, but we use a different conformal factor:

$$g \;=\; \frac{1}{(x^4)^2} g_E. \tag{5}$$

This determines a complete metric on the half-space $x^4 > 0$. To use our conformal change equations with $u = (x^4)^{-1}$, we compute the auxiliary tensor and its trace

$$\begin{aligned} K_{ij} \;&=\; -\frac{1}{2}\left(x^4\right)^{-2} \delta_{ij} \\ Tr(K) \;&=\; -2\left(x^4\right)^{-2}. \end{aligned} \tag{6}$$

The conformal change formulas give

$$\begin{aligned} \mathrm{Rm} \;&=\; \left(x^4\right)^{-2} (K \owedge g_E) \;=\; -\frac{1}{2}\left(x^4\right)^{-4} \delta \owedge \delta \;=\; -\frac{1}{2} g \owedge g \\ \mathrm{Ric} \;&=\; Tr(K) g \,+\, 2K \;=\; -3\left(x^4\right)^{-2} \delta \;=\; -3g \\ R \;&=\; -12 \end{aligned} \tag{7}$$

This is a metric of constant curvature -1.

Exercises

1. Given constants α, β, γ, show that the metric

$$g \;=\; \frac{4\alpha^2}{(\beta^2 - \gamma^2 r^2)^2} g_E \tag{8}$$

gives a metric constant negative curvature $-\frac{\alpha}{\beta\gamma}$.

(Updated Oct 2018)