Compact 4-metrics from squashed spheres

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1.1 $\mathbb{S}^4$

Among the many ways of expressing this metric, we write

$$g = dr^2 + \sin^2(r)\left(\eta_X^2 + \eta_Y^2 + \eta_Z^2\right) \tag{1}$$

This metric is Einstein with Einstein constant 3 and scalar curvature 12. The Weyl curvatures are zero (the metric is conformally flat). This metric is not Kähler

1.2 $\mathbb{C}P^2$

Among the many ways to express this metric, we use

$$g = \frac{1}{\left(1+\frac{\Lambda}{6}r^2\right)^2}\left(dr^2 + r^2\eta_X^2\right) + \frac{r^2}{1+\frac{\Lambda}{6}r^2}\left(\eta_Y^2 + \eta_Z^2\right), \quad r \in [0,\infty). \tag{2}$$

This is a multiple of the Fubini-Study metric, described elsewhere. This metric can be considered a limit of the Burns metric with positive cosmological constant.

The scalar curvature of this metric is 4Λ. The trace-free Ricci tensor is zero. This metric is Einstein with Einstein consant Λ, and is half-conformally flat. The Weyl curvatures are

$$W^+ = \frac{\Lambda}{6}\left(3\omega\otimes\omega - 2Id_{\Lambda^+}\right), \quad W^- = 0 \tag{3}$$

where ω is the Kähler form.

1.3 The Page metric on $\mathbb{C}P^2\sharp\overline{\mathbb{C}P}^2$

To express the Page metric we use the Taub-NUT-deSitter rubric

$$g \;=\; \frac{(1-\frac{4}{3}\Lambda n^2)(r^2-n^2)}{r^2+n^2-\frac{1}{3}\Lambda(r^2-n^2)^2}dr^2+4(r^2-n^2)\left(\eta_X^2+\eta_Y^2\right)+16n^2\frac{r^2+n^2-\frac{1}{3}\Lambda(r^2-n^2)^2}{(1-\frac{4}{3}\Lambda n^2)(r^2-n^2)}\eta_Z^2 \quad (4)$$

and $r\in(r_+,r_{++})$ where r_+, r_{++} are given below. These are Einstein with constant $\frac{\Lambda}{1-\frac{4}{3}\Lambda n^2}$. When this is positive, the metric is compact. To be smooth, two quantization conditions must be met. Factoring the expression $-\frac{1}{3}\Lambda(r^2-n^2)^2+r^2+n^2$ into $-\frac{1}{3}\Lambda(r^2-(r_+)^2)(r^2-(r_{++})^2)$, we see this is positive when $r\in(r_+,r_{++})$. Near the smaller root r_+, we change the metric to $\rho=2\sqrt{r-r_+}$ and express the metric in the fiber direction by

$$g \;=\; C_1(\rho)\left[d\rho^2+\left(2n\frac{\frac{1}{3}\Lambda r_+((r_{++})^2-(r_+)^2)}{\left(1-\frac{4}{3}\Lambda n^2\right)((r_+)^2-n^2)}\right)^2\rho^2\left(1+O(\rho^2)\right)d\psi^2\right], \quad (5)$$

and near the larger root r_{++} we change to $\rho=2\sqrt{r_{++}-r}$ and get

$$g \;=\; C_2(\rho)\left[d\rho^2+\left(2n\frac{\frac{1}{3}\Lambda r_{++}((r_{++})^2-(r_+)^2)}{\left(1-\frac{4}{3}\Lambda n^2\right)((r_{++})^2-n^2)}\right)^2\rho^2\left(1+O(\rho^2)\right)d\psi^2\right]. \quad (6)$$

For the metric to be smooth at both r_+ and r_{++}, we require the coefficients on the $\rho^2(1+O(\rho^2))d\psi^2$ to be k^2 where k is some integer. Further, they must be the *same* integer. We arrive at the quantization conditions

$$n\frac{\frac{1}{3}\Lambda r_+((r_{++})^2-(r_+)^2)}{\left(1-\frac{4}{3}\Lambda n^2\right)((r_+)^2-n^2)} \;=\; n\frac{\frac{1}{3}\Lambda r_{++}((r_{++})^2-(r_+)^2)}{\left(1-\frac{4}{3}\Lambda n^2\right)((r_{++})^2-n^2)} \;=\; \pm\frac{k}{2}. \quad (7)$$

One desires values of Λ and n for each k. However, as Page discovered, this is not possible, and another step is required: the Page limit. Page set $k=1$ and took a limit as Λ approaches its maximum value of $\frac{3}{4n^2}$. Choosing n appropriately, even though 4 becomes singular, one can change coordinates in the limit to obtain

$$\begin{aligned} g \;&=\; U^{-1}dr^2+4\frac{1-\nu^2\cos^2(r)}{3+6\nu^2-\nu^4}\left({\eta_X}^2+{\eta_Y}^2\right)+\frac{\sin^2(r)}{(3+\nu^2)^2}U{\eta_Z}^2, \\ U \;&=\; \frac{3-\nu^2-\nu^2(1+\nu^2)\cos^2(r)}{1-\nu^2\cos^2(r)}. \end{aligned} \quad (8)$$

This is the Page metric provided $\nu^4+4\nu^3-6\nu^2+12\nu-3=0$; this metric is Einstein only for this value. The Weyl tensors are recorded elsewhere in these notes.

References

[1] E. Calabi. "Extremal Kähler metrics." *In Seminar on differential geometry*, vol. 102, pp. 259-290. 1982.

[2] D. Page. "A compact rotating gravitational instanton." *Physics Letters B* 79 no 235 (1978)