Conformal Changing of Metrics

Primer

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1 Basics

Definition. Two Riemannian metrics on a common manifold are said to be conformally equivalent if, at each point, one is a scalar multiple of the other.

Usually there is a differentiability criterion, meaning that the conformal factor at each point must vary in a smooth way.

In studying conformally related metrics, one usually starts with a suitable well-known metric with understandable properties, say the flat metric on $\mathbb{R}^n$ or perhaps an ALE half-conformally flat metric, and multiplies it by a function chosen to make the metric interesting in some new way.

If g is the original metric, one typically obtains the new metric $\hat{g}$ in one of three ways. The first and most straightforward is

$$\hat{g} \;=\; u^2\, g \tag{1}$$

where $u > 0$ is a smooth function; but the formulas for here tend to involve terms with logarithms. Second, one may choose a conformal factor in exponential form

$$\hat{g} \;=\; e^{2f}\, g \tag{2}$$

where f is any smooth function and very possibly zero or negative in places. Third

$$\hat{g} \;=\; \psi^{\frac{4}{n-2}} g \tag{3}$$

where $\psi > 0$. This simplifies certain expressions, though clearly cannot be used in dimension 2. See 2.3 and 3.7.

We will give formulas for all three methods.

2 Conformal change formulas

Of great convenience will be the *Kulkarni Nomizu product*; see §2 of the general primer. In indices this is

$$(A \owedge B)_{ijkl} \;=\; A_{il}B_{jk} + A_{jk}A_{il} - A_{ik}B_{jl} - A_{jl}B_{ik}. \tag{4}$$

The product $A \owedge B$ is a 4-index tensor that has the same symmetries as the Riemann tensor, including the first Bianchi identity.

For $\widehat{g} = u^2 g$ the main conformal change formula is

$$\widehat{\mathrm{Rm}} \;=\; u^2\left(\mathrm{Rm} + K \owedge g\right) \tag{5}$$

where K is an appropriate tensor formed from u, and the form of the Riemann tensor we are using is $\mathrm{Rm} = \mathrm{Rm}_{ijkl}$.

This also gives a starting point for the study of conformal invariance.

Recall the notion of the *space of abstract Riemann tensors* (general primer §2); this is the vector space of tensors of the form $\bigwedge^2 \odot \bigwedge^2$ that obey the first Bianchi identity.

Given a metric, this space decopmoses into three subspaces:

- $\mathcal{R} = span_{\mathbb{R}}\left\{g \owedge g\right\}$, the space of abstract scalar curvatures
- $\mathcal{RC} = span_{\mathbb{R}}\left\{A \owedge g \;\middle|\; A \text{ is symmetric and trace-free }\right\}$, the space of abstract scalar curvatures
- Everything else: $\mathcal{W}$, the orthogonal complement to $\mathcal{R}$ and $\mathcal{RC}$, called the space of abstract Weyl tensors.

Any given Weyl tensor W, lying within the space of abstract Weyl tensors, is orthogonal to the conformal change tensor $K \owedge g$, and therefore W is unaffected by conformal change.

In other words W changes invariantly (some use the term *covariantly*) with conformal change: $W = u^2 W$ where we are using $W = W_{ijk}{}^{l}$.

2.1 First method: $\hat{g} = u^2 g$

It shall be convenient to use the auxiliary tensor

$$\begin{aligned} K &= -Hess(\log u) + d\log u \otimes df - \frac{1}{2}|\nabla \log u|^2 g \\ K_{ij} &= -(\log u)_{ij} + (\log u)_i (\log u)_j - \frac{1}{2}|\nabla \log u|^2 g_{ij} \end{aligned} \tag{6}$$

The Riemann tensor:

$$\begin{aligned} \widehat{\mathrm{Rm}} &= u^2 \left(\mathrm{Rm} + K \owedge g\right) \\ \widehat{\mathrm{Rm}}_{ijkl} &= e^{2f} \left(\mathrm{Rm}_{\,ijkl} + K_{il}g_{jk} + K_{jk}g_{il} - K_{ik}g_{jl} - K_{jl}g_{ik}\right) \end{aligned} \tag{7}$$

The Ricci tensor:

$$\begin{aligned} \widehat{\mathrm{Ric}} &= \mathrm{Ric} + Tr(K)g + (n-2)K \\ &= \mathrm{Ric} - (\triangle \log u)\, g \\ &\qquad + (n-2)\left(-Hess(\log u) + d\log u \otimes d\log u - |\nabla\ \log u|^2 g\right) \\ \widehat{\mathrm{Ric}}_{jk} &= \mathrm{Ric}_{\,ij} + (g^{st}K_{st})g_{ij} + (n-2)K_{ij} \\ &= \mathrm{Ric}_{\,ij} - (\triangle \log u)g_{ij} \\ &\qquad + (n-2)\left(-(\log u)_{ij} + (\log u)_i(\log u)_j - |\nabla \log u|^2 g_{ij}\right) \end{aligned} \tag{8}$$

The Scalar curvature:

$$\widehat{R} = u^{-2}\left(R - 2(n-1)\triangle \log u - (n-1)(n-2)|\nabla \log u|^2\right) \tag{9}$$

The Hessian: If φ is any function then

$$\begin{aligned} \widehat{Hess}(\varphi) &= Hess(\varphi) - (d\varphi \otimes d\log u + d\log u \otimes d\varphi) + \langle \nabla\varphi,\, \nabla \log u\rangle\ g_{ij} \\ \widehat{Hess}(\varphi)_{ij} &= Hess(\varphi)_{ij} - (\varphi_i(\log u)_j + (\log u)_i\varphi_j) + \left(g^{st}\varphi_s()\log u)_t\right)\ g_{ij} \end{aligned} \tag{10}$$

The Laplacian: If φ is any function then

$$\widehat{\triangle}\varphi = u^{-2}\left(\triangle\varphi + (n-2)\langle \nabla\varphi, \nabla \log u\rangle\right) \tag{11}$$

2.2 Second method: $\hat{g} = e^{2f} g$

It shall be convenient to use the auxiliary tensor

$$\begin{aligned} K &= -Hess(f) + df \otimes df - \frac{1}{2}|\nabla f|^2 g \\ K_{ij} &= -f_{ij} + f_i f_j - \frac{1}{2}|\nabla f|^2 g_{ij} \end{aligned} \tag{12}$$

The Riemann tensor:

$$\begin{aligned} \widehat{\mathrm{Rm}} &= e^{2f} \left(\mathrm{Rm} + K \owedge g \right) \\ \widehat{\mathrm{Rm}}_{ijkl} &= e^{2f} \left(\mathrm{Rm}_{ijkl} + K_{il} g_{jk} + K_{jk} g_{il} - K_{ik} g_{jl} - K_{jl} g_{ik} \right) \end{aligned} \tag{13}$$

The Ricci tensor:

$$\begin{aligned} \widehat{\mathrm{Ric}} &= \mathrm{Ric} + Tr(K) g + (n-2) K \\ &= \mathrm{Ric} - (\triangle f)\, g + (n-2) \left(-Hess(f) + df \otimes df - |\nabla f|^2 g \right) \\ \\ \widehat{\mathrm{Ric}}_{jk} &= \mathrm{Ric}_{ij} + (g^{st} K_{st}) g_{ij} + (n-2) K_{ij} \\ &= \mathrm{Ric}_{ij} - (\triangle f) g_{ij} + (n-2) \left(-f_{ij} + f_i f_j - |\nabla f|^2 g_{ij} \right) \end{aligned} \tag{14}$$

The Scalar curvature:

$$\widehat{R} = e^{-2f} \left(R - 2(n-1) \triangle f - (n-1)(n-2) |\nabla f|^2 \right) \tag{15}$$

The Hessian: If φ is any function then

$$\begin{aligned} \widehat{Hess}(\varphi) &= Hess(\varphi) - (d\varphi \otimes df + df \otimes d\varphi) + \langle \nabla \varphi, \nabla f \rangle \; g_{ij} \\ \widehat{Hess}(\varphi)_{ij} &= Hess(\varphi)_{ij} - (\varphi_i f_j + f_i \varphi_j) + \left(g^{st} \varphi_s f_t \right) \; g_{ij} \end{aligned} \tag{16}$$

The Laplacian: If φ is any function then

$$\widehat{\triangle} \varphi = e^{-2f} \left(\triangle \varphi + (n-2) \langle \nabla \varphi, \nabla f \rangle \right) \tag{17}$$

2.3 Third method: $\hat{g} = \psi^{\frac{4}{n-2}} g$

The two formulas that are simplified are the scalar curvature and the Laplacian. We have

$$\begin{aligned} \widehat{R} \;&=\; \psi^{-\frac{4}{n-2}} \left(R \;-\; 4\frac{n-1}{n-2}\psi^{-1}\triangle v \right) \\ &=\; \psi^{-\frac{n+2}{n-2}} \left(R\psi \;-\; 4\frac{n-1}{n-2}\triangle v \right) \\ \widehat{\triangle}\varphi \;&=\; \psi^{-\frac{4}{n-2}} \left(\triangle v \;+\; 2\left\langle \nabla\varphi,\, \nabla \log \psi \right\rangle \right) \end{aligned} \tag{18}$$

Also see the discussion on the conformal Laplacian in 3.7.

3 Conformal invariants

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3.1 The Hodge-star

The easiest of the conformal invariants is the Hodge-star operator.

Definition. The *volume form* is the totally anti-symmetric n-tensor, usually denoted $dVol$, given at the point p by

$$dVol_p\,(v_1,\,v_2,\ldots,v_n)\ =\ \pm\sqrt{det\ \left(\langle v_i,v_j\rangle\right)_{i,j=1}^n} \tag{19}$$

where $v_1,\ldots,v_n\in\,T_pM^n$.

With $\hat{g}\ =\ u^2g$ The volume form has conformal property

$$\widehat{dVol}\ =\ u^n dVol. \tag{20}$$

The matrix $\left(\langle v_i,v_j\rangle\right)_{i,j=1}^n$ is the symmetric $n\times n$ matrix of all inner products among the vectors v_i.

As discussed in §3 of the gneral primer, the arbitrary choice of $\pm$ is an expression of the choice of *orientation* on a manifold.

Definition. On an oriented manifold, the Hodge-star $*^{(k)}$ is the $(k,n-k)$ mixed tensor form of the volume tensor.

In particular $dVol=*^{(n)}$. Expressing the volume form in coordinates as

$$dVol\ =\ dVol_{i_1i_2\ldots i_n}\,dx^{i_1}\otimes dx^{i_2}\otimes\cdots\otimes dx^{i_n} \tag{21}$$

then the Hodge-star $*^{(k)}$ is given by

$$\begin{aligned} *^{\,(k)}\ &=\ {*^{(k)}}_{i_1\ldots i_k}{}^{i_{k+1}\ldots i_n}\,dx^{i_1}\otimes\ldots\otimes dx^{i_k}\otimes\frac{\partial}{\partial x^{i_{k+1}}}\otimes\cdots\otimes\frac{\partial}{\partial x^{i_n}},\quad\text{where}\\ {*^{(k)}}_{i_1\ldots i_k}{}^{i_{k+1}\ldots i_n}\ &=\ dVol_{i_1\ldots i_kj_{k+1}\ldots j_n}g^{i_{k+1}j_{j+1}}\ldots g^{i_nj_n}. \end{aligned} \tag{22}$$

Recall the convention of how a tensor like $*^{(k)}$ acts: If $A=A_{i_1\ldots i_k}dx^{i_1}\otimes\cdots\otimes dx^{i_k}$ is a k-form, then $*^{(k)}A$ is the $(n-k)$-form

$$\left(*^{(k)}A\right)\ =\ \frac{1}{k!}\left({*^{(k)}}_{i_1\ldots i_k}{}^{j_1\ldots j_{n-k}}A_{i_1\ldots i_k}\right)dx^{j_1}\otimes\cdots\otimes dx^{i_{n-k}} \tag{23}$$

Notice the coefficient.

With $\hat{g}=u^2g$ the Hodge-star has conformal property

$$\widehat{*^{(k)}}\ =\ u^{2k-n}\ *^{(k)}\,. \tag{24}$$

In particular if the manifold M^n is even-dimensional, then $\widehat{*^{(n/2)}}=*^{(n/2)}$.

3.2 The Weyl tensor

The primary conformal invariant is the *Weyl Tensor* W.

Definition. The *Weyl Tensor* is defined to be the tensor

$$W \triangleq \mathrm{Rm} - \frac{R}{2n(n-1)} g \owedge g - \frac{1}{n-2} \mathring{\mathrm{Ric}} \owedge g. \tag{25}$$

Coming to grips with W is often tough for beginners in geometry; but see §2 of the general primer, where we discuss W at length.

For now, only recall that it is easily shown that the tensors

$$R\, g \owedge g, \qquad \mathring{\mathrm{Ric}} \owedge g, \qquad W \tag{26}$$

are mutually orthogonal to each other. The decomposition

$$\mathrm{Rm} = \frac{R}{2n(n-1)} g \owedge g + \frac{1}{n-2} \mathring{\mathrm{Ric}} \owedge g + W \tag{27}$$

is, in fact, an orthogonal decomposition of the Riemann tensor; this is known as the *Ricci decomposition* of the Riemann tensor.

In dimension 3, the Weyl tensor always vanishes—all curvature information is contained in the scalar curvature and the trace-free Ricci tensor.

In dimension 2 both the trace-free Ricci tensor and the Weyl tensor vanish—all curvature information is contained in the scalar curvature, which is twice the Gaussian curvature in this case.

With $\hat{g} = u^2 g$ and using the fact that $\widehat{\mathrm{Rm}} = u^2 \left(\mathrm{Rm} + K \owedge g\right)$ and W is orthogonal to anything of the form $K \owedge g$, we see that the Weyl tensor is a conformal invariant:

$$\begin{aligned} \widehat{W}_{ijkl} &= u^2 W_{ijkl} \\ \widehat{W}_{ijk}{}^{l} &= W_{ijk}{}^{l} \end{aligned} \tag{28}$$

3.3 The special case of 4 dimensions

Four-dimensional manifolds are special for many reasons; among them is the fact that the Weyl tensor decomposes into two pieces.

In dimension 4, the operator $*^{(2)} = *^{(2)}{}_{ij}{}^{kl}$ takes anti-symmetric 2-forms to anti-symmetric 2-forms, and one can prove that

$$\frac{1}{2}\, *^{(2)\ kl}{}_{st}\, *^{(2)\ st}{}_{ij} \;=\; Id^{kl}{}_{ij} \tag{29}$$

(where Id is the identity operator: $\frac{1}{2}Id^{kl}{}_{ij}A_{kl} = A_{ij}$ for any anti-symmetric tensor A—note the choice of factor, which is the convention (23)). That is, $*^{(2)}$ squares to the identity, and so as an operator from antisymmetric 2-tensors to antisymmetric 2-tensors, $*^{(2)}$ has eigenvalues of $+1$ and -1. We can define corresponding projection operators

$$\begin{aligned} \pi_+ &\triangleq \frac{1}{2}\left(Id + *^{(2)}\right) \\ \pi_- &\triangleq \frac{1}{2}\left(Id - *^{(2)}\right). \end{aligned} \tag{30}$$

Certainly $\pi_\pm \circ \pi_\pm = \pi_\pm$ and $\pi_\pm \circ \pi_\mp = 0$ and $\pi_\pm + \pi_\mp = Id$, so that π_+, π_- are complementary idempotents and therefore projectors.

Using these projectors, in dimension 4 we can split the Weyl tensor into two mutually orthogonal pieces:

$$\begin{aligned} W^+_{ijkl} &= \frac{1}{4} W_{stuv}\, \pi_+{}^{uv}{}_{kl}\, \pi_+{}^{st}{}_{ij} \\ W^-_{ijkl} &= \frac{1}{4} W_{stuv}\, \pi_-{}^{uv}{}_{kl}\, \pi^{st}{}_{ij} \end{aligned} \tag{31}$$

(the $\frac{1}{4}$ comes from the convention (23)). One may verify $W_{stuv}\pi_-{}^{uv}{}_{kl}\pi_+{}^{st}{}_{ij} = W_{stuv}\pi_+{}^{uv}{}_{kl}\pi_-{}^{st}{}_{ij} = 0$ and so therefore

$$W = W^+ + W^-. \tag{32}$$

In four dimensions we have the extended Ricci decomposition

$$\mathrm{Rm} \;=\; \frac{R}{2n(n-1)}\, g \owedge g \;+\; \frac{1}{n-2}\, \mathring{\mathrm{Ric}} \owedge g \;+\; W^+ \;+\; W^-. \tag{33}$$

Each piece $W^\pm$ is conformally invariant: $\widehat{W^\pm} = u^2 W^\pm$ where $\widehat{g} = u^2 g$.

3.4 The Cotton tensor and the special case of 3 dimensions

Definition. The *Schouten tensor* is

$$P_{ij} \;=\; \frac{1}{n-2}\left(\mathrm{Ric}_{\,ij} \;-\; \frac{R}{2(n-1)}g_{ij}\right) \tag{34}$$

This tensor P is defined so that $\mathrm{Rm} = P \owedge g + W$; in other words, the Schouten tensor is effectively the sum of the scalar and trace-free Ricci components of the Riemann tensor.

Definition. The *Cotton tensor* is

$$C_{ijk} \;=\; P_{ij,k} \;-\; P_{ik,j}. \tag{35}$$

The Cotton tensor is defined for the following purpose: the divergence of Weyl is given by

$$W^{s}{}_{ijk,s} \;=\; (n-3)C_{ijk}. \tag{36}$$

In dimension 3 this is clearly vacuous, as $W = 0$ and $n - 3 = 0$. But the Cotton tensor does not vanish in dimension 3.

With $\hat{g} = u^2 g$ and using the auxiliary tensor $K_{ij} = -\left(\log\, u\right)_{ij} + \left(\log\, u\right)_i \left(\log\, u\right)_j - \frac{1}{2}|\nabla \log u|^2 g_{ij}$, the conformal properties of P and C are

$$\begin{aligned} \widehat{P} \;&=\; P \;+\; K \\ \widehat{P}_{ij} \;&=\; P_{ij} \;+\; K_{ij} \\ \\ \widehat{C} \;&=\; C \;+\; W\left(\nabla \log u,\,\cdot,\,\cdot,\,\cdot\right) \\ \widehat{C}_{ijk} \;&=\; C_{ijk} \;+\; W^{s}{}_{ijk}(\log u)_s. \end{aligned} \tag{37}$$

In dimension 3, clearly the Cotton tensor is a conformal invariant.

3.5 The special case of 2 dimensions

The uniqueness of two dimensional geometry can be traced to three phenomena:

i) The Riemann tensor is completely determined by the scalar curvature.

ii) The Laplacian is a conformal invariant.

iii) The Hodge-star on covectors is a complex structure: $*^{(1)}*^{(1)} = -Id$.

From a less pedestrian viewpoint, (*i*) can be traced back to the fact that the irreducible representations of $\mathfrak{so}(2)$ are quite trivial, and (*ii*) and (*iii*) ultimately trace to the fact that $SO(2) = U(1)$.

First, letting G, resp. $\widehat{G}$, be the Gaussian curvature of g, resp. $\hat{g} = u^2 g$, we have

$$\begin{aligned} \widehat{G} &= u^{-2}\left(G - \triangle \log u\right) \\ \widehat{R} &= u^{-2}\left(R - 2\triangle \log u\right) \\ \widehat{\triangle}\varphi &= u^{-2}\triangle\varphi \end{aligned} \tag{38}$$

Scond, letting G, resp. $\widehat{G}$, be the Gaussian curvature of g, resp. $\hat{g} = e^{2f} g$, we have

$$\begin{aligned} \widehat{G} &= e^{-2f}\left(G - \triangle f\right) \\ \widehat{R} &= e^{-2f}\left(R - 2\triangle f\right) \\ \widehat{\triangle}\varphi &= e^{-2f}\triangle\varphi \end{aligned} \tag{39}$$

3.6 The Bach tensor

Definition. The *Bach tensor* is

$$B_{ij} \;=\; W^s{}_{ij}{}^t{}_{,st} \;+\; \frac{1}{2}\,\mathrm{Ric}\,_{st} W^s{}_{ij}{}^t. \tag{40}$$

The Bach tensor is the L^2 gradient tensor of the Weyl functional, in the following sense. If $g_t = g + t\dot{g} + O(t^2)$ is a time-dependent variation of the metric and if the manifold M^n is compact, then

$$\left.\frac{d}{dt}\right|_{t=0} \int_{M^n} |W|^2 \, dVol \;=\; \int \dot{g}^{ij} B_{ij} \, dVol. \tag{41}$$

The Bach tensor is trace-free: $g^{ij} B_{ij} = 0$.

The Bach tensor is divergence-free: $B^s{}_{j,s} = 0$.

The Bach tensor is conformally invariant: $\widehat{B}_{ij} \;=\; B_{ij}$.

3.7 The conformal Laplacian

The conformal Laplacian is defined to be

$$L\varphi \;=\; \triangle\varphi \,-\, \frac{1}{4}\frac{n-2}{n-1}R\varphi. \tag{42}$$

This is "conformally covariant," in the sense that if $\hat{g} = \psi^{\frac{4}{n-2}}g$, then

$$\widehat{L}\left(\psi^{-1}\varphi\right) \;=\; \psi^{-\frac{n+2}{n-2}}L\varphi \tag{43}$$

and we have the convenient formula

$$\widehat{R} \;=\; -\frac{4(n-1)}{n-2}\psi^{-\frac{n+2}{n-2}}L\psi. \tag{44}$$

in all dimensions except 2.

Other conformal differential operators exist, for instance the fourth order Paneitz operator and the higher order GJMS operators. To keep the discussion fairly basic, we do not go into these.

4 Exercises

1. Recall an exercise from the general primer that $Id : \bigwedge^2 \to \bigwedge^2$ is $Id^{kl}{}_{ij} = \frac{1}{2}(g \owedge g)_{ijst} g^{sl} g^{tk}$. Show that this identity is $\delta^l_i \delta^k_i - \delta^k_i \delta^l_j$.

   On a 4-manifold we have also $*^{(2)} : \bigwedge^2 \to \bigwedge^2$. Can you express this in terms of Kroenecker δ-functions?

2. Verify that W is always orthogonal to $A \owedge g$ for any symmetric 2-tensor A. Conclude that (27) is indeed an orthogonal decomposition.

3. Using $\hat{g} = u^2 g$, prove formally $\widehat{W}_{ijkl} = u^2 W_{ijkl}$.

4. In dimension 4, prove directly $*^{(2)}{}_{ij}{}^{st} \, *^{(2)}{}_{st}{}^{kl} = Id_{ij}{}^{kl}$ (also written $(*^{(2)})^2 = Id$) on antisymmetric 2-tensors.

5. In dimension 4, prove that

$$
\begin{aligned}
(\mathring{\mathrm{Ric}} \owedge g)_{stuv} \, \pi_{+ij}{}^{st} \, \pi_{+kl}{}^{uv} = 0 \\
(\mathring{\mathrm{Ric}} \owedge g)_{stuv} \, \pi_{-ij}{}^{st} \, \pi_{-kl}{}^{uv} = 0.
\end{aligned}
\tag{45}
$$

6. In dimension 4, prove that W^+ and W^- are orthogonal to one another.

7. Using the conformal invariance of W, prove that $g^{ij} B_{ij} = 0$ that B is conformally invariant.

8. (Harder) Using the diffeomorphism invariance of Rm (and therefore of W), show that $B^s{}_{j,s} = 0$.

9. Verify formula (43).

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