Metrics on 4-dimensional Manifolds

Primer

1 Background

We will assume a fairly minimal level of background, but the purpose of this primer is not to be an introduction to Riemannian geometry. Some background in Riemannian geometry and smooth manifold theory is necessary, and some theory of complex geometry would certainly also help.

Roughly one should be more or less conversant in the following:

- Differentiable manifold theory, roughly equivalent to the first six chapters of Milnor's *Topology from the Differentiable Viewpoint* [7].
- Knowledge of the exterior algebra would be helpful, perhaps enough to understand chapter 1.A. of Besse's *Einstein Manifolds* [1]. Certainly the exterior derivative d should be understood.
- Differential geometry, roughly equivalent to the first seven chapters of do Carmo's *Riemannian Geometry* [3].
- Basics of complex manifold theory, roughly the main points of the first 3 chapters of Huybrecht's *Complex Geometry: An Introduction* [4] or perhaps two thirds of chapter 2 of *Einstein Manifolds* [1].

The purpose of this primer is first to establish the notation and conventions used throughout the rest of these notes, and second to run through some material that might be more specialized to four-dimensional geometry (largely for the benefit of students). Few theorems are proved. Although clarity is hoped for, the emphasis is on density of information. Topics are chunked into one-page summaries.

Some of the more specialized constructions may require some material not in these notes. Probably 80-90% of the examples will be understandable after digesting this primer.

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2 Riemannian geometry

Skip to a subsection:

2.1 Conventions

We lay out some conventions that, sadly, can vary from author to author.

Definition. The *Riemann tensor* is

$$\mathrm{Rm}(X,\,Y)Z \;=\; \nabla_X\nabla_Y Z \;-\; \nabla_Y\nabla_X Z \;-\; \nabla_{[X,Y]}Z \tag{1}$$

(this is sometimes called the anti-do Carmo convention).

Definition. The *Laplacian* of a function f is the trace of its Hessian: $\triangle f = g^{ij} f_{,ij}$.

Definition. If A is an antisymmetric k-tensor and B is an antisymmetric l-tensor, the *exterior product* of A and B is the antisymmetric $k+l$ tensor $A \wedge B$ defined by

$$\begin{aligned} &A \wedge B\,(v_1, \dots, v_{k+l}) \\ &\quad = \frac{1}{k!}\frac{1}{l!} \sum_{\pi \in Sym(k+l)} (-1)^{sgn\,\pi} A\left(v_{\pi(1)},\,\dots,\,v_{\pi(k)}\right) B\left(v_{\pi(k+1)},\,\dots,\,v_{\pi(k+l)}\right) \end{aligned} \tag{2}$$

Definition. If $A = A_{ij}dx^i \otimes dx^j$ and $B = B_{ij}dx^i \otimes dx^j$ are symmetric tensors, their *Kulkarni-Nomizu product* $A \owedge B = (A \owedge B)_{ijkl} dx^i \otimes dx^j \otimes dx^k \otimes dx^l$ is

$$\begin{aligned} (A \owedge B)(v_1, v_2, v_3, v_4) \;\triangleq\; & A(v_1,v_4)B(v_2,v_3) + A(v_2,v_3)B(v_1,v_4) \\ & - A(v_1,v_3)B(v_2,v_4) - A(v_2,v_4)B(v_1,v_3) \end{aligned} \tag{3}$$

In indices, $(A \owedge B)_{ijkl} = A_{il}B_{jk} + A_{jk}A_{il} - A_{ik}B_{jl} - A_{jl}B_{ik}$. The tensor $A \owedge B$ has the index symmetries of the Riemann tensor, including the first Bianchi identity.

Definition. If A and B are k-tensors, their *tensor inner product* is $\langle A,\,B\rangle_T = g^{i_1j_i} \dots g^{i_kj_k} A_{i_i \dots i_k} B_{j_i \dots j_k}$. If A and B are anti-symmetric (that is, they are k-forms), their *form inner product* is

$$\langle A,\,B\rangle_A = \frac{1}{k!}\,\langle A,\,B\rangle_T \tag{4}$$

This is defined so that, for instance, $|dVol|^2_T = 1$. For convenience we usually neglect the A or T subscript and let context decide.

Definition. The *musical isomorphisms* $\# : T^*M \to TM$, $\flat : TM \to T^*M$ are

$$\left\langle \eta^\#,\,\cdot\right\rangle \;=\; \eta, \quad v_\flat \;=\; \langle v,\,\cdot\rangle\,. \tag{5}$$

These are mutual inverses, also known as the "raising and lowering" of indices:

$$\eta^\# \;=\; \left(g^{ij}\eta_j\right)\frac{\partial}{\partial x^i}, \qquad v_\flat \;=\; \left(g_{ij}v^j\right)dx^i \tag{6}$$

2.2 Frames and the Cartan formalism

In addition to the Ricci formalism, where Christoffel symbols and tensors like $\mathrm{Rm}_{ijk}{}^{l}$ are computed in some coordinate patch, we have the Cartan formalism. Instead of coordinates, the basic object of the Cartan formalism is the frame.

The Cartan formalism tends to be less familiar to students, but in many contexts it is the far superior computational technique.

Definition. A *frame* on some domain Ω in a Riemannian manifold M^n is a set of n-many covector fields, denoted $\{\eta^1, \dots, \eta^n\}$, that form a complete orthonormal basis of T_p^*M at each point $p \in \Omega$.

Theorem. (Cartan's Lemma) Given any complete independent set of covector fields $\{\eta^i\}$, there is a unique matrix of 1-forms $\theta = (\theta^i_j)$ that satisfy $d\eta^i = -\theta^i_j \wedge \eta^j$ and $\theta^i_j = -\theta^j_i$.

Definition. Given a frame $\{\eta^i\}$ on a Riemannian manifold, the matrix $(\theta^i_j) = (\theta^i_{jk}\eta^k)$ of 1-forms given by Cartan's Lemma is called the frame's *connection 1-form.*

Definition. (Cartan) Given a frame $\{\eta^i\}$ on a Riemannian manifold, its *curvature 2-form* is

$$\Omega^i_j \;=\; d\theta^i_j \,+\, \theta^i_s \wedge \theta^s_j. \tag{7}$$

This is sometimes abbreviated $\Omega = d\theta + \theta \wedge \theta$, or indeed $\Omega = d\theta + \frac{1}{2}[\theta, \theta]$.

It can be proved that

$$\mathrm{Rm}_{ijk}{}^{l}\, dx^i \otimes dx^j \;=\; \Omega^l_k. \tag{8}$$

(Students should prove this as an exercise.)

The Ricci curvature tensor and the and scalar curvature can be computed within this formalism as well: The Ricci vector $\mathcal{RC} = (\mathrm{Ric}_k)_{k=1}^n$ is a vector of 1-forms

$$\mathrm{Ric}_k \;=\; i_{\eta^l}\Omega^l_k \tag{9}$$

and the scalar curvature is

$$R \;=\; \sum_k i_{\eta^k}\, \mathrm{Ric}_k. \tag{10}$$

2.3 Decomposition of the Riemann tensor

The Kulkarni-Nomizu product is a great too for pulling apart the Riemann tensor. First a few facts: if A, B are symmetric 2-tensors the tensor norms are

$$\begin{aligned} \langle A \owedge g,\, B \owedge g\rangle_T &= 4\left(\langle A, g\rangle_T \langle B, g\rangle_T + (n-2)\langle A, B\rangle_T\right), \\ \langle A \owedge g,\, g \owedge g\rangle_T &= 8(n-1)\langle A, g\rangle_T, \qquad |g \owedge g|_T^2 = 8n(n-1). \end{aligned} \tag{11}$$

Definition. The *trace* of a 2-tensor A is $Tr\, A \triangleq \langle A, g\rangle_T$.

Definition. The *trace-free Ricci tensor* is $\mathring{\mathrm{Ric}} = \mathrm{Ric} - \frac{R}{n} g$. (Notice $\langle \mathring{\mathrm{Ric}}, g\rangle = 0$.)

We examine some interactions with the curvature tensor. From (11) we see that $g \owedge g$ and $\mathring{\mathrm{Ric}} \owedge g$ are orthogonal to each other. Projecting the Riemann tensor onto the 1-dimensional spaces spanned by these basis elements, we easily compute

$$\begin{aligned} Proj_{g \owedge g}(\mathrm{Rm}) &= \left\langle \mathrm{Rm}, \frac{g \owedge g}{\sqrt{8n(n-1)}} \right\rangle \frac{g \owedge g}{\sqrt{8n(n-1)}} = \frac{R}{2n(n-1)} g \owedge g \\ Proj_{\mathring{\mathrm{Ric}} \owedge g}(\mathrm{Rm}) &= \left\langle \mathrm{Rm}, \frac{\mathring{\mathrm{Ric}} \owedge g}{2\sqrt{n-2}|\mathring{\mathrm{Ric}}|} \right\rangle \frac{\mathring{\mathrm{Ric}} \owedge g}{2\sqrt{n-2}|\mathring{\mathrm{Ric}}|} = \frac{1}{n-2} \mathring{\mathrm{Ric}} \owedge g. \end{aligned} \tag{12}$$

Definition. The *space of abstract Riemann tensors* is the vector space of all 4-component tensors with the symmetries of the Riemann tensor; in other words the subspace of $\bigwedge^2 \odot \bigwedge^2$ that obeys the first Bianchi identity; see §3.2 for information about the spaces $\bigwedge^k$.

Definition. The *Weyl tensor* is the projection of Rm on to the subspace perpendicular to $span\{g \owedge g, \mathring{\mathrm{Ric}} \owedge g\}$ within the space of abstract Riemann tensors. In other words $W = \mathrm{Rm} - \frac{R}{2n(n-1)} g \owedge g - \frac{1}{n-2} \mathring{\mathrm{Ric}} \owedge g$.

The Weyl tensor has zero Ricci trace: $W_{ijkl} g^{jk} = 0$. The orthogonal decomposition

$$\mathrm{Rm} = \frac{R}{2n(n-1)} g \owedge g + \frac{1}{n-2} \mathring{\mathrm{Ric}} \owedge g + W \tag{13}$$

is known as the *Ricci decomposition* of the curvature tensor.

2.4 Comments on the Weyl tensor

The definition of W usually seems opaque, even absurd, to most beginners. But it is a very interesting curvature-like object, and carries a great deal of useful, understandable information about a metric. However, unlike the scalar curvature R and Ricci curvature Ric, the Weyl curvature tensor can rarely be understood fully; it always retains a bit of mystery.

The first understanding of the Weyl tensor is as a kind of remainder: when the contribution of the scalar and Ricci curvatures are "subtracted" out of the Riemann tensor, everything that remains is the Weyl tensor.

In General Relativity, the Weyl Tensor has a meaning as the tidal curvature operator. Given a ball of non-interacting dust moving under gravitational influence, the Ricci tensor encodes its infinitesimal volume change acceleration. The Weyl tensor can be thought of as a kind of space-time shear. In part, the Weyl tensor encodes infinitesimal tidal acceleration; it also encodes something not readily sensible to most humans: frame-dragging. Specifically, in general Relativity W^+, called the *electrical* component of the Weyl tensor, encodes tidal acceleration, and W^-, called the *magnetic* component of the Weyl tensor, encodes frame-drag acceleration. See [6] for details. See §3.7 for the definitions of $W^\pm$.

In Riemannian geometry we can see some of this (though it is tough to directly make sense of an electrical/magnetic distinction). Due to conformal invariance, the Weyl tensor is insensitive to volumes, at least infinitesimally. If you consider the case of concentric geodesic balls expanding from a point, the Weyl tensor, at least infinitesimally, can't measure deviation from Euclidean volume growth. Rather, it encodes how oblong or elliptically shaped the balls are.

Sadly at long ranges this view does not quite fit, and the situation is quite a bit more complicated. There are well-known examples of Riemannian metrics on topological $\mathbb{R}^4$ with $R \equiv 0$, $\mathring{\mathrm{Ric}} \equiv 0$, and $W^+ \equiv 0$—so the only non-zero component of Riemannian curvature is W^-—but where volumes of large balls are $O(r^3)$ as their radii r gets big rather than $O(r^4)$ as would be expected if volume were Euclidean-like at long distances. The main example is the so-called Euclidean Taub-NUT metric.

In dimension 3, the Weyl tensor always vanishes—all curvature information is contained in the scalar curvature and the trace-free Ricci tensor.

In dimension 2 both trace-free Ricci and Weyl vanish—all curvature information is contained in the scalar curvature, which is twice the Gaussian curvature.

2.5 Actions of the curvature tensors

The Riemann tensor is antisymmetric in its first two entries and its last two entries, and symmetric between the first two indices and last two indices. In other words we can consider Rm to be a section

$$\mathrm{Rm} \;\in\; \bigwedge{}^2 \odot \bigwedge{}^2 \tag{14}$$

where $\odot$ is the symmetric product. This means we can dualize Rm to obtain a map

$$\mathrm{Rm} \;:\; \bigwedge{}^2 \;\longrightarrow\; \bigwedge{}^2 \tag{15}$$

which is the tensor form $\mathrm{Rm}^{ij}{}_{kl}$ of the Riemann tensor. However, recalling the sign convention for Rm and also the coefficient on the inner product $\langle\cdot,\cdot\rangle_A$ from (4), for any tensor $\mathcal{C} \in \bigwedge^2 \odot \bigwedge^2$ it is best to define

$$\mathcal{C}(\eta) \;=\; \frac{1}{2}\mathcal{C}^{ts}{}_{kl}\eta_{st}\, dx^k \otimes dx^l, \tag{16}$$

Note the s, t index reversal. The reason for this convention and for the coefficient of $\frac{1}{2}$ can be seen, for instance, in the fact that $\frac{1}{2}g \owedge g : \bigwedge^2 \to \bigwedge^2$ should be the identity. And indeed using $\eta_{ij} = -\eta_{ji}$ we have

$$\begin{aligned} \frac{1}{2}(g \owedge g)(\eta) \;&=\; \frac{1}{4}(g \owedge g)^{ts}{}_{ij}\eta_{st}dx^i \otimes dx^j \;=\; \frac{1}{4}(\delta^s_i\delta^t_j - \delta^t_j\delta^s_i)\eta_{st}dx^i \otimes dx^j \\ &=\; \frac{1}{2}(\eta_{ij} - \eta_{ji})dx^i \otimes dx^j \;=\; \eta. \end{aligned} \tag{17}$$

The components of Rm also act on 2-forms. Notably the Weyl action

$$W \;:\; \bigwedge{}^2 \longrightarrow \bigwedge{}^2 \tag{18}$$

is $W(\eta)_{ij} = \frac{1}{2}W^{ts}{}_{ij}\eta_{st}$. It can be shown that as a map $\bigwedge^2 \to \bigwedge^2$, $\mathring{\mathrm{Ric}} \owedge g$ has the property of being trace-free:

$$\begin{aligned} \mathring{\mathrm{Ric}} \owedge g \;&:\; \bigwedge{}^2 \longrightarrow \bigwedge{}^2, \\ Tr\,(\mathring{\mathrm{Ric}} \owedge g) \;&=\; 0. \end{aligned} \tag{19}$$

In four dimensions these maps, particularly W, have additional special properties; see §3.7 for these properties.

3 The exterior algebra and the Hodge-star

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3.1 The Hodge Star

Definition. The *volume form* is the totally anti-symmetric n-tensor, usually denoted $dVol$, given at the point p by

$$dVol_p\,(v_1,\,\ldots,\,v_n) \;=\; \pm\sqrt{det\;\left(\langle v_i, v_j\rangle\right)_{i,j=1}^{n}} \tag{20}$$

where $v_1, \ldots, v_n \in T_pM^n$.

The matrix $\left(\langle v_i, v_j\rangle\right)_{i,j=1}^{n}$ is the symmetric $n \times n$ matrix of all inner products among the vectors v_i. In simple cases such as on 2- or 3-dimensional manifolds, one easily verifies that $dVol$ gives the signed area of the parallelogram spanned by v_1, v_2 or the volume of the parallelepiped spanned by v_1, v_2, v_3, respectively. In higher dimensions, it gives the hypervolume of the parallelotope spanned by $v_1, \ldots, v_n$.

One immediately notices a problem: the arbitrary choice of $\pm$. A manifold is *orientable* if this choice can be made consistently on every coordinate patch over the entire manifold, and *non-orientable* otherwise. An *orientation* for an orientable manifold is a consistent choice of $+$ or $-$ on every coordinate patch. Any connected, orientable manifold always has precisely two orientations.

Definition. On an oriented manifold, the Hodge-star $*^{(k)}$ is the $(k, n-k)$ mixed tensor form of the volume tensor.

In particular $dVol = *^{(n)}$. Expressing the volume form in coordinates as

$$\begin{aligned} dVol \;&=\; dVol_{i_1 i_2 \ldots i_n}\, dx^{i_1} \otimes dx^{i_2} \otimes \cdots \otimes dx^{i_n} \\ &=\; \frac{1}{n!} dVol_{i_1 i_2 \ldots i_n}\, dx^{i_1} \wedge dx^{i_2} \wedge \cdots \wedge dx^{i_n} \end{aligned} \tag{21}$$

then the Hodge-star $*^{(k)}$ is given by

$$\begin{aligned} *^{\,(k)} \;&=\; *^{(k) i_1 \ldots i_k}{}_{i_{k+1} \ldots i_n}\, \frac{\partial}{\partial x^{i_1}} \otimes \cdots \otimes \frac{\partial}{\partial x^{i_k}} \otimes dx^{i_{k+1}} \otimes \ldots \otimes dx^{i_n}, \quad \text{where} \\ *^{(k)\; i_1 \ldots i_k}{}_{i_{k+1} \ldots i_n} \;&=\; g^{i_1 j_1} \ldots g^{i_k j_k} dVol_{j_1 \ldots j_k i_{k+1} \ldots i_n}. \end{aligned} \tag{22}$$

3.2 The exterior algebra

Definition. The *exterior algebra* over a point p in a manifold, denoted $\bigwedge_p^*$ or just $\bigwedge^*$, is the algebra of all antisymmetric tensors, under the exterior product.

Certainly $\bigwedge^*$ is a graded algebra:

$$\bigwedge{}^* \;=\; \bigwedge{}^0 \otimes \bigwedge{}^1 \otimes \ldots \otimes \bigwedge{}^n \tag{23}$$

where each $\bigwedge^i$ is the $\mathbb{R}$-vector space of dimension $\binom{n}{i}$ spanned by forms $dx^{i_1} \wedge \cdots \wedge dx^{i_k}$. Clearly also $dim \bigwedge{}^* = n!$.

The vector space $\bigwedge^0$ is the 1-dimensional space of coefficients; a section of $\bigwedge^0$ is a function. The vector space $\bigwedge^1$ is the n-dimensional space of covectors; a section of $\bigwedge^1$ is a covector field.

Since the vector space $\bigwedge^n$ is also 1-dimensional, $\bigwedge^n$ and $\bigwedge^0$ are abstractly isomorphic. But they also have an inner product—on $\bigwedge^n$ this is $\langle \cdot, \cdot \rangle_A$ and on $\bigwedge^0$ this is just multiplying two numbers. Thus exactly two choices of isometric isomorphisms exist that identify $\bigwedge^n$ and $\bigwedge^0$; choosing one of these consistently at all points of the manifold is the same choosing an orientation, as described in (3.1).

A differential operator $d : \bigwedge{}^k \to \bigwedge{}^{k+1}$ exists, and is defined purely in terms of the manifold's differentiable structure. If $A \in \bigwedge{}^k$ then $dA \in \bigwedge{}^{k+1}$ is defined by

$$\begin{aligned} (dA)\,(v_0, v_1, \ldots, v_n) \;&=\; \sum_{i=0}^{n} (-1)^i v_i \left(A(v_0, \ldots, \widehat{v_i}, \ldots, v_n)\right) \\ &+ \sum_{i<j}^{n} (-1)^{i+j} A([v_i, v_j],\, v_0, \ldots, \widehat{v_i}, \ldots, \widehat{v_j}, \ldots, v_n) \end{aligned} \tag{24}$$

Parallel to this definition, the exterior differential can also be defined as the unique operator $d : \bigwedge^k \to \bigwedge^{k+1}$ satisfying

i) Constant-linearity: $d(c_1\eta_1 + c_2\eta_2) = c_1 d\eta_1 + c_2 d\eta_2$ for $c_1, c_2 \in \mathbb{R}$, $\eta_1, \eta_2 \in \bigwedge{}^*$

ii) Leibniz Rule: If $\eta \in \bigwedge^k$ and $\mu \in \bigwedge^l$ then $d\,(\eta \wedge \mu) = d\eta \wedge \mu + (-1)^k \eta \wedge d\mu$.

iii) Closure rule: $ddf = 0$.

iv) Function rule: if $f \in \bigwedge^0$ is a function and v a vector field, then $(df)(v) = v(f)$.

One can show that $dd\eta = 0$ for any k-form η (exercise).

3.3 The exterior algebra and the Hodge-star

The Hodge-star acts on any k-form to produce an $(n-k)$-form. If A is any anti-symmetric k-tensor so we may write

$$A = A_{i_1 \ldots i_k} dx^{i_1} \otimes \cdots \otimes dx^{i_k} = \frac{1}{k!} A_{i_1 \ldots i_k} dx^{i_1} \wedge \cdots \wedge dx^{i_k}, \tag{25}$$

then we define

$$\begin{aligned} *^{(k)} : \bigwedge{}^{k} &\longrightarrow \bigwedge{}^{n-k} \\ \left(*^{(k)} A\right)_{j_1 \ldots j_{n-k}} &= \frac{1}{k!} *^{(k)\ i_1 \ldots i_k}{}_{j_1 \ldots j_{n-k}} A_{i_1 \ldots i_k} \end{aligned} \tag{26}$$

Notice that this is not strictly a tensor contraction, but requires multiplication by a factor of $(k!)^{-1}$. This convention is chosen so that the following nice formula holds:

$$\begin{aligned} *^{(k)} *^{(n-k)} : \bigwedge{}^{k} &\longrightarrow \bigwedge{}^{k} \\ *^{(k)} *^{(n-k)} &= (-1)^{k(n-k)}. \end{aligned} \tag{27}$$

The Hodge-star can be used as an alternative way to define inner products on members of $\bigwedge^{*}$. If $A, B \in \bigwedge^{k}$ then the form inner product is

$$\langle A,\, B \rangle_A = *^{(n)} \left(A \wedge *^{(k)} B\right). \tag{28}$$

This formula is another reason for the normalizing factors in (2), (4), and (27).

Equation (28) means $*^{(k)}$ can be considered a duality operator; specifically, the dual of $A \in \bigwedge^{k}$ can be considered the operator $*^{(n)}\left(\cdot \wedge *^{(k)} A\right) : \bigwedge^{k} \to \mathbb{R}$. The map $*^{(k)} : \bigwedge^{k} \to \bigwedge^{n-k}$ can be considered a duality isometric isomorphism.

We often use $* : \bigwedge^{k} \to \bigwedge^{n-k}$ in place $*^{(k)} : \bigwedge^{k} \to \bigwedge^{n-k}$ and use context to distinguish which subspace $\bigwedge^{k}$ is being acted on.

The "Hodge-star" is often called the "Hodge-*" or the "Hodge duality operator."

3.4 Dual Operators

The Hodge-star, due to its function as a duality operator, can be used to define the so-called *codifferential*, and the so-called *interior product* via dualization.

Definition. The *exterior codifferential* $\delta : \bigwedge^k \to \bigwedge^{k-1}$ is the operator

$$\delta \;=\; (-1)^{nk+n+1} \, * \, d \, * \tag{29}$$

The operator δ unlike d relies on the existence of a Riemannian metric. There is a reason why δ should be considered the dual of d; to discuss this we must introduce the L^2 inner product.

Definition. If $\eta_1, \eta_2 \in \bigwedge^k$, then the L^2 *inner product* of η_1 and η_2 is defined to be

$$\langle \eta_1, \, \eta_2 \rangle_{L^2} \;=\; \int_{M^n} \eta_1 \wedge \, *^{(k)} \, \eta_2. \tag{30}$$

Note that $\langle \eta_1, \eta_2 \rangle_{L_2} = \int \langle \eta_1, \eta_2 \rangle_A \, dVol.$

Exercise. The codifferential δ is the L^2-dual of the differential d:

$$\langle d\eta_1, \, \eta_2 \rangle_{L^2} \;=\; \langle \eta_1, \, \delta\eta_2 \rangle_{L^2} \, . \tag{31}$$

This is the reason for the strange sign in (29).

Although the view of δ as the dual of d is tremendously useful, there is a standard list of troublesome aspects here: eg what if M^n has a boundary; what if M^n is open; d is not a continuous linear operator, so technically it doesn't have a dual; etc.

Definition. If $\eta \in \bigwedge^k$, $\mu \in \bigwedge^l$ and $k \leq l$, the *interior product* $i_\eta \mu \in \bigwedge^{l-k}$ is

$$i_\eta \mu \;=\; *^{(n+k-l)} \left(\eta \wedge *^{(l)} \mu \right) . \tag{32}$$

Given any $\eta \in \bigwedge^k$, the operator $i_\eta : \bigwedge^l \to \bigwedge^{l-k}$ is a derivation of order $-k$ (see §3.5 below for the definition of "derivation"). Like d, i_η has an axiomatic definition:

i) Constant-linearity: $i_\eta(c_1\mu_1 + c_2\mu_2) = c_1 i_\eta \mu_1 + c_2 i_\eta \mu_2$ for $c_1, c_2 \in \mathbb{R}$, $\mu_1, \mu_2 \in \bigwedge^*$.

ii) Leibniz Rule: If $\mu \in \bigwedge^l$, $\gamma \in \bigwedge^m$ then $i_\eta (\mu \wedge \gamma) = i_\eta \mu \wedge \gamma + (-1)^{kl} \mu \wedge i_\eta \gamma$

iii) Closure rule: $i_\eta i_\eta \mu = 0$.

iv) Inner product rule: if $\mu \in \bigwedge^k$ then $i_\eta \mu = \langle \eta, \mu \rangle_A$.

3.5 Derivations and the Lie derivative

We have seen a similar set of axioms twice, once for the exterior derivative and once for the interior product. This can be formalized into the notion of a derivation.

Definition. A *derivation of order* k is any map $\mathcal{D} : \bigwedge^l \to \bigwedge^{k+l}$ that obeys

i) Constant-linearity: $\mathcal{D}(c_1 A\eta_1 + c_2\eta_2) = c_1\mathcal{D}\eta_1 + c_2\mathcal{D}\eta_2$ for $c_1, c_2 \in \mathbb{R}$.

ii) Leibniz Rule: If $\eta \in \bigwedge^l$, $\mu \in \bigwedge^m$ then $\mathcal{D}\left(\eta \wedge \mu\right) = \mathcal{D}\eta \wedge \mu + (-1)^{kl}\eta \wedge \mathcal{D}\mu$.

The exterior derivative d and the interior product i_η are derivations.

Definition. If $\mathcal{D}_1$, $\mathcal{D}_2$ are derivations of orders k_1, k_2, respectively, then the *derivation bracket* of $\mathcal{D}_1$, $\mathcal{D}_2$ is

$$[\mathcal{D}_1,\, \mathcal{D}_2] \;=\; \mathcal{D}_1\mathcal{D}_2 \,-\, (-1)^{k_1k_2}\mathcal{D}_2\mathcal{D}_1. \tag{33}$$

Exercise. Verify that the derivation bracket is itself a derivation of order $k_1 + k_2$.

Definition. Given $\eta \in \bigwedge^k$, the *Lie derivative* in the "direction" η is

$$\mathcal{L}_\eta \;=\; [d,\, i_\eta]. \tag{34}$$

This is used most often when $\eta \in \bigwedge^1$ is a covector, in which case it makes sense to say the Lie derivative is a derivative in some "direction," namely the direction $\eta^\#$ (see section §**??** for the definition of $\#$). If $\eta \in \bigwedge^1$ then

$$\mathcal{L}_\eta \;=\; di_\eta \,+\, i_\eta d. \tag{35}$$

The Lie derivative has a remarkable interpretation. If $v \in TM^n$ is the vector field that is metric dual to η, and ψ_t is the diffeomorphism flow associated to v, then

$$\mathcal{L}_\eta\mu \;=\; \frac{d}{dt}\bigg|_{t=0}\psi_t^*(\mu). \tag{36}$$

The Lie derivative does *not* obey the closure axiom: generally $\mathcal{L}_\eta\mathcal{L}_\eta \neq 0$. Interactions between $\mathcal{L}_\eta$ other derivations are frequently useful: if $\eta, \mu \in \bigwedge^1$ then

$$[\mathcal{L}_\eta,\, d] \;=\; 0, \quad [\mathcal{L}_\eta,\, \mathcal{L}_\mu] \;=\; \mathcal{L}_{[\eta,\mu]}, \quad [\mathcal{L}_\eta,\, i_\mu] \;=\; i_{[\eta,\mu]}, \quad [i_\eta,\, i_\mu] \;=\; 0. \tag{37}$$

3.6 Laplacians

Three Laplacians typically come up in differential geometry: the "rough" Laplacian, the Hodge Laplacian, and the Lichnerowicz Laplacian (which we won't discuss). In complex geometry there are three further Laplacians; see §4.8.

Definition. If $A = A_{i_1 \dots i_k}$ is any tensor, its *rough Laplacian* is

$$\triangle A \;=\; g^{jk} A_{i_1 \dots i_k,\, jk}. \tag{38}$$

(Note our new convention here: we write $A_{i_1 \dots i_k}$ in place of $A_{i_1 \dots i_k} dx^{i_1} \otimes \cdots \otimes dx^{i_k}$.) One can prove that $\triangle A = -\nabla^* \nabla A$ where ∇^* is the L^2-dual of the Levi-Civita connection ∇. Again we're ignoring the difficulties, one being that $\nabla : \bigotimes^k T^*M \to TM \otimes \bigotimes^k T^*M$ is not a bounded linear operator so technically has no dual.

Definition If $\eta \in \bigwedge^k$ is any k-form, its *Hodge Laplacian* is the k-form

$$\triangle_d \eta \;=\; dd^*\eta + d^*d\eta \;=\; [d, d^*]\eta. \tag{39}$$

(Although we've invoked the derivation bracket, d^* is not a derivation and neither is $\triangle_d$.) The operators $\triangle_d$ and $-\triangle$ have the same second-order parts; this is not tough to show by direct computation. In fact that they have the same first order parts also, and only differ in their constant terms. If $\eta \in \bigwedge^k$, it can be shown

$$-\triangle \eta \;=\; \triangle_d \eta \;+\; dx^i \wedge \left(i_{dx^j} \left(\mathrm{Rm} \left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j} \right) \eta \right) \right) \tag{40}$$

where we have used $\mathrm{Rm}(\partial_i, \partial_j)\eta = \nabla_{\partial_j} \nabla_{\partial_i} \eta - \nabla_{\partial_i} \nabla_{\partial_j} \eta$ for any tensor η. This is the *Bochner formula* in general. Some individual Bochner formulas are

$$\begin{aligned} -\triangle \eta \;&=\; \triangle_d \eta + \mathrm{Ric}(\eta^{\#}, \cdot) && \text{for} \quad \eta \in \textstyle\bigwedge^1, \\ -\triangle \eta \;&=\; \triangle_d \eta + \frac{2}{n}\frac{n-2}{n-1} R\eta + \frac{n-4}{n-2}\mathrm{Ric}(\eta) - 2W(\eta) && \text{for} \quad \eta \in \textstyle\bigwedge^2, \end{aligned} \tag{41}$$

where W is Weyl curvature (see §2.3, §2.4). For $\eta \in \bigwedge^2$ the conventions are

$$\begin{aligned} \mathrm{Ric}(\eta) &= \mathrm{Ric}\,{}_i{}^s \eta_{sj}\, dx^i \wedge dx^j \quad \in \textstyle\bigwedge^2 \\ W(\eta) &= \frac{1}{2} W^{ts}{}_{ij} \eta_{st}\, dx^i \otimes dx^j \quad \in \textstyle\bigwedge^2. \end{aligned} \tag{42}$$

Notice the index reversal and the coefficient; see §2.5 for the rationale behind this, and also for more on W and Rm as maps of the type $\bigwedge^2 \to \bigwedge^2$.

3.7 The Hodge-star on 4-manifolds

The Hodge-star on 4-manifolds has certain unique features. In dimension 4, the operator $*^{(2)} : \bigwedge^2 \to \bigwedge^2$ takes anti-symmetric 2-forms to antisymmetric 2-forms, and by (27) we have

$$*^{(2)}*^{(2)} = Id : \bigwedge{}^2 \to \bigwedge{}^2. \tag{43}$$

That is, $*^{(2)}$ is a root of unity and so has eigenvalues of $+1$ and -1. The space $\bigwedge^2$ is six dimensional, and the three dimensional subspaces $\bigwedge^+$, $\bigwedge^-$ are

$$\begin{aligned} \bigwedge{}^+ &= \left\{\eta \in \bigwedge{}^2 \mid *\eta = \eta\right\} \\ \bigwedge{}^- &= \left\{\eta \in \bigwedge{}^2 \mid *\eta = -\eta\right\}. \end{aligned} \tag{44}$$

We define corresponding projection operators

$$\begin{aligned} \pi_+ &\triangleq \frac{1}{2}(Id + *) : \bigwedge{}^2 \to \bigwedge{}^+ \\ \pi_- &\triangleq \frac{1}{2}(Id - *) : \bigwedge{}^2 \to \bigwedge{}^-. \end{aligned} \tag{45}$$

Certainly $\pi_\pm \circ \pi_\pm = \pi_\pm$ and $\pi_\pm \circ \pi_\mp = 0$ and $\pi_\pm + \pi_\mp = Id$, so π_+, π_- are complementary idempotents and therefore a complete set of orthogonal projectors. Using these projectors, we can split the curvature operator into four parts:

$$\mathrm{Rm}^{\pm\pm} = \frac{1}{4}\mathrm{Rm}_{st}{}^{uv}\, \pi_\pm{}^{st}{}_{ij}\, \pi_\pm{}^{kl}{}_{uv}, \quad \mathrm{Rm}^{\pm\pm} : \bigwedge{}^\pm \longrightarrow \bigwedge{}^\pm. \tag{46}$$

(the rationale for the factor of $\frac{1}{4}$ is in §2.1, §2.5, and §3.3) Each projection Rm^{++}, Rm^{+-}, ... is easily computed, and Rm in block form is

$$\mathrm{Rm} = \left(\begin{array}{c|c} \mathrm{Rm}^{++} & \mathrm{Rm}^{-+} \\ \hline \mathrm{Rm}^{+-} & \mathrm{Rm}^{--} \end{array}\right) = \left(\begin{array}{c|c} \frac{R}{24}(g \owedge g)\big|_{\bigwedge^+} + W^+ & \frac{1}{2}\mathring{\mathrm{Ric}} \owedge g \\ \hline \frac{1}{2}\mathring{\mathrm{Ric}} \owedge g & \frac{R}{24}(g \owedge g)\big|_{\bigwedge^-} + W^- \end{array}\right) \tag{47}$$

Defining W^{+-}, etc, similarly as Rm^{+-}, etc, we find that $W^{+-} = W^{-+} = 0$ and it is typical to abbreviate $W^+ = W^{++}$, $W^- = W^{--}$. In four dimensions we have the extended Ricci decomposition

$$\mathrm{Rm} = \frac{R}{24} g \owedge g + \frac{1}{2}\mathring{\mathrm{Ric}} \owedge g + W^+ + W^-. \tag{48}$$

3.8 Curvature and the Hodge-star

Consider the curvature 2-form $\Omega = (\Omega_i^j)$, which is a matrix of 2-forms. We use $\Omega \wedge \Omega$ for matrix multiplication *cum* exterior product:

$$(\Omega \wedge \Omega)_j^i \;\triangleq \Omega_s^i \wedge \Omega_j^s. \tag{49}$$

Thus $\Omega \wedge \Omega$ is a matrix of 4-forms. Of course we can apply the trace operator to any matrix: since Ω is anti-symmetric $Tr\,\Omega = 0$; but $Tr\,\Omega \wedge \Omega = \Omega_s^t \wedge \Omega_t^s$ is a 4-form.

The Riemann tensor is a section of $\bigwedge^2 \otimes \bigwedge^2$, so the Hodge-star can work on Rm in two different ways; we denote $*\,\mathrm{Rm} \in \bigwedge^{n-2} \otimes \bigwedge^2$ and $\mathrm{Rm}\,* \in \bigwedge^2 \otimes \bigwedge^{n-2}$, and of course $*\,\mathrm{Rm}\,* \in \bigwedge^{n-2} \otimes \bigwedge^{n-2}$.

In the context of Ω_j^i, one of these $*$ operators is easy to use: we have that $*\Omega = (*\Omega_j^i)$ is a matrix of $(n-2)$-forms obtained by simply applying $*$ to each entry. We have for instance that $\Omega \wedge *\Omega$ is a matrix of n-forms. Starting from the standard matrix inner product $\langle A, B\rangle = Tr(AB^T)$ and noting that Ω and $*\Omega$ are antisymmetric, we have the norm-square of the curvature tensor:

$$|\Omega|^2 \;=\; Tr * \left(\Omega \wedge (*\Omega)^T\right) \;=\; - * \left(\Omega_s^t \wedge *\Omega_t^s\right). \tag{50}$$

Using the Hodge-star in the second way (above denoted $\mathrm{Rm}\,*$) is a bit less convenient when one is dealing with the curvature 2-form Ω. This is because the Hodge-star has an easy definition when dealing with 2-forms $\eta = \eta_{ij}$ but not when dealing with antisymmetric matrices $A = A_j^i$.

In 4-dimensions this changes, however. The $*$-operator takes 2-forms to 2-forms, so we should be able to express it as an operator that takes antisymmetric matrices to antisymmetric matrices. We denote this transformation by $\star$, the Pfaffian operator.

Definition. The *Pfaffian operator* $\star$ is a linear operator that changes a 4×4 antisymmetric matric to another 4×4 antisymmetric matrix by

$$A \;=\; \begin{pmatrix} 0 & A_2^1 & A_3^1 & A_4^1 \\ & 0 & A_3^2 & A_4^2 \\ & & 0 & A_4^3 \\ & & & 0 \end{pmatrix} \;\longmapsto\; A\star \;=\; \begin{pmatrix} 0 & A_4^3 & -A_4^2 & A_3^2 \\ & 0 & A_4^1 & -A_3^1 \\ & & 0 & A_2^1 \\ & & & 0 \end{pmatrix} \tag{51}$$

Definition. The *Pfaffian* of an antisymmetric matrix is

$$Pf(A) \;=\; \frac{1}{4} Tr\left(A \cdot (A\star)^T\right) \tag{52}$$

On any oriented 4-manifold, the curvature Pfaffian is $Pf(\Omega) = -\frac{1}{4} * (\Omega_s^t \wedge *(\Omega\star)_t^s)$.

3.9 Characteristic Classes

We consider here only 4-manifolds, and we delve deeper into the effect of the two Hodge-star operations $*\,\mathrm{Rm} = *\Omega$ and $\mathrm{Rm}\,\star = \Omega\star$ (recall $\star$ is the Pfaffian operator).

Recalling the form decomposition $\bigwedge^2 = \bigwedge^+ \oplus \bigwedge^-$ and the decomposition (47) of the Riemann tensor in dimension 4, we have decompositions

$$
\begin{aligned}
*\,\mathrm{Rm} &= \left(\begin{array}{c|c} \frac{R}{24}(g \owedge g)\big|_{\bigwedge^+} + W^+ & \frac{1}{2}\,\mathring{\mathrm{Ric}} \owedge g \\ \hline -\frac{1}{2}\,\mathring{\mathrm{Ric}} \owedge g & -\left(\frac{R}{24}(g \owedge g)\big|_{\bigwedge^-} + W^-\right) \end{array}\right), \\
\mathrm{Rm}\,\star &= \left(\begin{array}{c|c} \frac{R}{24}(g \owedge g)\big|_{\bigwedge^+} + W^+ & -\frac{1}{2}\,\mathring{\mathrm{Ric}} \owedge g \\ \hline \frac{1}{2}\,\mathring{\mathrm{Ric}} \owedge g & -\left(\frac{R}{24}(g \owedge g)\big|_{\bigwedge^-} + W^-\right) \end{array}\right), \qquad (53) \\
*\,\mathrm{Rm}\,\star &= \left(\begin{array}{c|c} \frac{R}{24}(g \owedge g)\big|_{\bigwedge^+} + W^+ & -\frac{1}{2}\,\mathring{\mathrm{Ric}} \owedge g \\ \hline -\frac{1}{2}\,\mathring{\mathrm{Ric}} \owedge g & \frac{R}{24}(g \owedge g)\big|_{\bigwedge^-} + W^- \end{array}\right).
\end{aligned}
$$

where $*\,\mathrm{Rm}, \mathrm{Rm}\,\star, *\,\mathrm{Rm}\,\star$ are operators $\bigwedge^2 \to \bigwedge^2$. From (53) we readily observe

$$
\begin{aligned}
|\,\mathrm{Rm}\,|^2 &= Tr_{\bigwedge^2}(\mathrm{Rm}\circ\mathrm{Rm}) \;=\; -Tr *(\Omega\wedge *\Omega) \\
&= \frac{R^2}{6} + 2|\,\mathring{\mathrm{Ric}}\,|^2 + |W^+|^2 + |W^-|^2, \\
\langle \mathrm{Rm}, *\,\mathrm{Rm}\rangle &= Tr_{\bigwedge^2}(\mathrm{Rm}\circ *\,\mathrm{Rm}) \;=\; -Tr *(\Omega\wedge\Omega) \\
&= |W^+|^2 - |W^-|^2, \qquad (54) \\
Pf(\mathrm{Rm}) &= \frac{1}{4}Tr(\mathrm{Rm}\circ *\,\mathrm{Rm}\,*) \;=\; -\frac{1}{4}Tr *(\Omega\wedge *\Omega\star) \\
&= \frac{R^2}{24} - \frac{1}{2}|\,\mathring{\mathrm{Ric}}\,|^2 + \frac{1}{4}|W^+|^2 + \frac{1}{4}|W^-|^2.
\end{aligned}
$$

For reasons we won't go into (but see Kobayashi-Nomizu [5]), the deRham classes of the 4-forms $\Omega\wedge\Omega$ and $*Pf(\Omega)$ carry topological information. A compact 4-manifold M^4 has signature $\tau(M^4) = \frac{1}{12\pi^2}\int \frac{1}{4}\Omega\wedge\Omega$ and Euler number $\chi(M^4) = \frac{1}{8\pi^2}\int *Pf(\Omega)$. These are topological numbers, so unlike $\int *|\,\mathrm{Rm}\,|^2$, do not change with the metric.

4 Complex Geometry

Skip to a subsection:

4.1 Almost Complex Structures

Definition. An *almost complex structure* is a smoothly varying map $J : T_pM \to T_pM$ so that squares to minus the identity: $J(J(v)) = -v$ for all $v \in T_pM$.

We can extend J to act on covectors by

$$J(\eta) \;=\; \eta \circ J, \qquad \text{for} \quad \eta \in \bigwedge{}^1. \tag{55}$$

(Some authors use $J(\eta) = -\eta \circ J$.) Since $J^2 = -Id$, the operator J has eigenvalues of $\pm\sqrt{-1}$, and so in order to decompose T_pM into eigenspaces we must complexify.

Definition. The complexified tangent space is $T_p^{\mathbb{C}}M = T_pM \otimes \mathbb{C}$. The complexified tangent bundle is the union $T^{\mathbb{C}}M = \bigcup_p T_p^{\mathbb{C}}M$.

Complex vectors are simply ordinary vectors with complex coefficients. We have complexified cotangent spaces and a complexified cotangent bundle $\bigwedge^1_{\mathbb{C}} \triangleq \bigwedge^1 \otimes \mathbb{C}$.

Definition. The *holomorphic* and *anti-holomorphic* tangent spaces, respectively, and cotangent spaces, respectively are

$$\begin{gathered} T_p'M = \left\{ v \in T^{\mathbb{C}}M \;\middle|\; Jv = \sqrt{-1}v \right\}, \; T_p''M = \left\{ v \in T^{\mathbb{C}}M \;\middle|\; Jv = -\sqrt{-1}v \right\}, \\ \bigwedge{}^{1,0} = \left\{ \eta \in \bigwedge{}^1_{\mathbb{C}}M \;\middle|\; J\eta = \sqrt{-1}\eta \right\}, \; \bigwedge{}^{0,1} = \left\{ \eta \in \bigwedge{}^1_{\mathbb{C}}M \;\middle|\; J\eta = -\sqrt{-1}\eta \right\}. \end{gathered} \tag{56}$$

$T'M$ and $\bigwedge^{1,0}$ are duals: $\eta(v) = 0$ when $\eta \in \bigwedge^{1,0}$ and $v \in T''M$ and vice versa. The entire exterior algebra can be complexified: $\bigwedge^{*,*} = \bigwedge^* \otimes \mathbb{C}$.

Definition. The vector space $\bigwedge^{p,q}$ is the span of exterior products of p many $\bigwedge^{1,0}$ forms and q many $\bigwedge^{0,1}$ forms.

This gives what is called the *bidegree decomposition* of the spaces $\bigwedge^k \otimes \mathbb{C}$ for $k \in \{0, \dots, n\}$. It can be proved that

$$\bigwedge{}^k \otimes \mathbb{C} \;=\; \bigwedge{}^{k,0} \oplus \bigwedge{}^{k-1,1} \oplus \cdots \oplus \bigwedge{}^{0,k} \;=\; \bigoplus\nolimits_{i=0}^{k} \bigwedge{}^{k-i,i}. \tag{57}$$

Almost complex structures can only exist on even dimensional spaces. On any chart, it is possible to choose $2n$ many coordinates $(x^1, y^1, \dots, x^n, y^n)$ where $Jdx^i = -dy^i$. Defining complex-valued functions $z^i = x^i + \sqrt{-1}y^i$, we have

$$span_{\mathbb{C}} \left\{ dz^1, \dots, dz^n \right\} \;=\; \bigwedge{}^{1,0}, \quad span_{\mathbb{C}} \left\{ d\bar{z}^1, \dots, d\bar{z}^n \right\} \;=\; \bigwedge{}^{0,1}. \tag{58}$$

Dual to dz^i is the field $\frac{\partial}{\partial z^i} \triangleq \frac{1}{2}\left(\frac{\partial}{\partial x^i} - \sqrt{-1}\frac{\partial}{\partial y^i} \right)$, and such fields span $T'M$. Likewise the fields $\frac{\partial}{\partial \bar{z}^i} \triangleq \frac{1}{2}\left(\frac{\partial}{\partial x^i} + \sqrt{-1}\frac{\partial}{\partial y^i} \right)$ are dual to the $d\bar{z}^i$ and span $T''M$.

4.2 Complex Structures

Definition. A *holomorphic structure* on a manifold M^{2n} is a maximal set of charts $\{\psi_\alpha : U_\alpha \to \mathbb{C}^n\}$ so that the transitions $\psi_\alpha \circ \psi_\beta : \psi_\beta(U_\beta) \cap \psi_\alpha(U_\alpha) \to \psi_\beta(U_\beta) \cap \psi_\alpha(U_\alpha)$ are holomorphic

A manifold with a holomorphic structure is called a *complex manifold*. On $\mathbb{C}$ there is a natural almost complex structure $J(\partial/\partial x^i) = \partial/\partial y^i$, $J(\partial/\partial y^i) = -\partial/\partial x^i$. Any holomorphic map F on $\mathbb{C}^n$ commutes with J: $dF \circ J = J \circ dF$. Therefore any complex structure induces an almost complex structure.

Definition. An almost complex structure J is called a *complex structure* if it is induced by a holomorphic structure on M^{2n}.

Definition. Given J, $N_J(X,Y) \triangleq [JX,JY] - [X,Y] - J[JX,Y] - J[X,JY]$.

This is called the *Nijenhuis tensor*. With an almost complex structure we have operators ∂ and $\bar{\partial}$ given by

$$\begin{aligned} \partial &: \bigwedge^{p,q} \to \bigwedge^{p+1,q}, \quad \partial = \pi^{p+1,q} d \\ \bar{\partial} &: \bigwedge^{p,q} \to \bigwedge^{p+1,q}, \quad \bar{\partial} = \pi^{p,q+1} d \end{aligned} \tag{59}$$

where $\pi^{i,j} : \bigwedge_{\mathbb{C}}^{i+j} \to \bigwedge^{i,j}$ are the projectors. Normally $\partial\partial \neq 0$ and $\bar{\partial}\bar{\partial} \neq 0$.

Theorem. An almost complex structure J is a complex structure iff any of the following equivalent criteria hold:

i) $N_J = 0$

ii) If $v \in T'M$ and $w \in T'M$ then $[v,w] \in T'M$

iii) $\partial\partial = 0$ or $\bar{\partial}\bar{\partial} = 0$

iv) $d = \partial + \bar{\partial}$

v) $d : \bigwedge^{p,q} \to \bigwedge^{p+1,q} \oplus \bigwedge^{p,q+1}$

vi) $d : \bigwedge^{0,1} \to \bigwedge^{1,1} \oplus \bigwedge^{0,2}$.

This is a deep theorem. That (*i*) is on the list is called the Newlander-Nirenburg theorem. Because of (*ii*), an almost complex structure is called *integrable* if is a complex structure; this is by analogy with the classic Frobenius theorems. Because of (*iii*) and (*iv*) we have $\partial\bar{\partial} = -\bar{\partial}\partial$ for complex structures.

4.3 Hermitian metrics

Definition. If J is an almost complex structure and g a Riemannian metric, then J is called *compatible with the metric* if $g(X,Y) = g(JX,JY)$, all $X, Y \in T^{\mathbb{C}}M$.

Definition. If J is compatible with g, then $\omega(X,Y) \triangleq g(JX,Y)$ is an antisymmetric 2-tensor called the *Kähler form* associated to g, J.

Definition. An *Hermitian metric* is any operator $\langle\cdot,\,\cdot\rangle : T'M \otimes T'M \to \mathbb{C}$ that satisfies

—(Sesquilinearity) $\langle cX_1 + X_2, Y\rangle = c\,\langle X_1, Y\rangle + \langle X_2, Y\rangle$ and $\langle X, cY_1 + Y_2\rangle = \bar{c}\,\langle X, Y_1\rangle + \langle X, Y_2\rangle$
—(Skew Symmetry) $\langle X, Y\rangle \;=\; \overline{\langle Y, X\rangle}$
—(Positivity) $\langle X, X\rangle \geq 0$ with equality iff $X = 0$.

Although similar to a metric, an Hermitian metric is not a tensor since it is sesquilinear, not linear. Still an Hermitian metric can be expressed in tensor form, as long as its action includes an artificial use of complex conjugation.

A tensor $h = h_{\alpha\bar{\beta}} dz^\alpha \otimes d\bar{z}^\beta$ that has $\overline{h_{\alpha\bar{\beta}}} = h_{\beta\bar{\alpha}}$ and so the matrix $(h_{\alpha\bar{\beta}})$ has all positive eigenvalues, then an Hermitian metric is

$$\langle X,\, Y\rangle \;=\; h\left(X,\, \overline{Y}\right). \tag{60}$$

An Hermitian metric is not a metric, but in fact is equivalent to having a metric with compatible almost complex structure. Indeed

$$\begin{aligned} h \;&=\; g + \sqrt{-1}\omega \\ g \;&=\; Sym(h) \;=\; \frac{1}{2} h_{\alpha\bar{\beta}} \left(dz^\alpha \otimes d\bar{z}^\beta + d\bar{z}^\beta \otimes dz^\alpha\right) \\ \omega \;&=\; \sqrt{-1}Alt(h) \;=\; \frac{\sqrt{-1}}{2} h_{\alpha\bar{\beta}} \left(dz^\alpha \wedge d\bar{z}^\beta\right). \end{aligned} \tag{61}$$

Using h we have duality isomorphisms and skew-isomorphisms:

$$v_\flat \;=\; h_{i\bar{j}} v^i d\bar{z}^j, \quad \eta^\sharp \;=\; h^{i\bar{j}} \eta_i \frac{\partial}{\partial \bar{z}^i}, \quad v_{\bar{\flat}} \;=\; h_{i\bar{j}} \overline{v^j} dz^i, \quad \eta^{\bar{\sharp}} \;=\; h^{i\bar{j}} \overline{\eta_j} \frac{\partial}{\partial z^j} \tag{62}$$

where $(h^{i\bar{j}})$ is the matrix inverse of $(h_{i\bar{j}})$. The linear (resp. skew) versions of these maps are $\flat : T'M \to \bigwedge^{0,1}$, $\sharp : \bigwedge^{1,0} \to T''_{\mathbb{C}}M$ (resp. $\bar{\flat} : T'M \to \bigwedge^{1,0}$, $\bar{\sharp} : \bigwedge^{1,0} \to T'_{\mathbb{C}}M$).

A first order form of compatibility can be imposed, which ties the metric more directly to the complex structure.

Definition. Assuming J is integrable (so M^{2n} is a complex manifold) then ω satisfies the *Kähler condition* and (M^{2n}, J, ω) is a *Kähler manifold* if $d\omega \;=\; 0$.

4.4 Chern Connections

On a complex manifold we have a closed $\bar\partial$ operator $\bar\partial : \bigwedge^{p,q} \to \bigwedge^{p,q+1}$. In addition to this, we can actually expand $\bar\partial$ to an operator $\bar\partial : T'M \to \bigwedge^{0,1} \otimes T'M$.

Definition. On a chart $(z^1, \dots, z^n)$, the operator $\bar\partial$ is

$$\bar\partial Z \;\triangleq\; \frac{\partial c^i}{\partial \bar z^j} d\bar z^j \otimes \frac{\partial}{\partial z^i} \qquad \text{where} \qquad Z = c^i \frac{\partial}{\partial z^i}. \tag{63}$$

To see that this definition is chart-independent, let $(z^1, \dots, z^n)$, $(w^1, \dots, w^n)$ be coordinates on overlapping charts; then $\frac{\partial z^i}{\partial \bar w^j} = 0$ and $\frac{\partial w^i}{\partial \bar z^j} = 0$ and we compute

$$\begin{aligned}\bar\partial\left(c^i \frac{\partial}{\partial z^i}\right) \;&=\; \frac{\partial c^i}{\partial \bar z^j} d\bar z^j \otimes \frac{\partial}{\partial z^i} \;=\; \left(\frac{\partial \bar w^k}{\partial \bar z^j}\frac{\partial c^i}{\partial \bar w^k}\right)\left(\frac{\partial \bar z^j}{\partial \bar w^s} d\bar w^s\right) \otimes \left(\frac{\partial w^t}{\partial z^i}\frac{\partial}{\partial w^t}\right) \\ &=\; \frac{\partial}{\partial \bar w^k}\left(c^i \frac{\partial w^t}{\partial z^i}\right) d\bar w^k \otimes \frac{\partial}{\partial w^t} \;=\; \bar\partial\left(c^i \frac{\partial w^t}{\partial z^i}\frac{\partial}{\partial w^t}\right).\end{aligned} \tag{64}$$

Thus, due to the holomorphicity of the transitions, this definition is chart-invariant. We can extend $\bar\partial$ to

$$\bar\partial : \bigwedge^{p,q} \otimes T'M \longrightarrow \bigwedge^{p,q+1} \otimes T'M \tag{65}$$

by forcing the Leibniz rule:

$$\bar\partial\left(\eta \otimes Z\right) \;=\; \bar\partial\eta \otimes Z \;+\; (-1)^{p+q} \eta \wedge \bar\partial Z. \tag{66}$$

Definition. A connection ∇ is called *compatible with the Hermitian metric* if

$$d\left\langle X,\, Y\right\rangle \;=\; \left\langle \nabla X,\, Y\right\rangle \;+\; \left\langle X,\, \nabla Y\right\rangle \tag{67}$$

where we regard a connection as a map $\nabla : T'M \to \bigwedge^1_{\mathbb{C}} \otimes T'M$. Similarly to $\bar\partial$, any connection can be extended to a map $\nabla : \bigwedge^k_{\mathbb{C}} \otimes T'M \to \bigwedge^{k+1}_{\mathbb{C}} \otimes T'M$ by forcing the Leibniz rule:

$$\nabla\left(\eta \otimes X\right) \;=\; d\eta \otimes X \;+\; (-1)^k \eta \wedge \nabla X. \tag{68}$$

Sometimes this extended connection is called a *covariant exterior derivative.*

Definition. A connection ∇ on a complex manifold is called a *Chern connection* if it is compatible with the metric, and if its $(0,1)$-part is just $\bar\partial$; that is

$$\nabla X \;-\; \bar\partial X \;\;\in\;\; \bigwedge^{1,0} \otimes T'M. \tag{69}$$

Often we use the matrix $\theta^i_j \in \bigwedge^{1,0}$ for the Christoffel symbols: since $\bar\partial \frac{\partial}{\partial z^j} = 0$ we have $\nabla \frac{\partial}{\partial z^j} \in \bigwedge^{1,0} \otimes T'M$ so we can write $\nabla \frac{\partial}{\partial z^j} \;=\; \theta^i_j \otimes \frac{\partial}{\partial z^i}$, where $\theta^i_j = \theta^i_{js} dz^s$.

4.5 The Chern curvature

Theorem. Chern connections are unique. Indeed with $h_{i\bar{j}} = \left\langle \frac{\partial}{\partial z^i}, \frac{\partial}{\partial z^j} \right\rangle$ we have

$$\theta^i_j \;=\; h^{i\bar{k}} \partial h_{j\bar{k}} \tag{70}$$

Definition. $\Omega = \nabla^2$, $\Omega : \bigwedge^k \otimes T'M \rightarrow \bigwedge^{k+2} \otimes T'M$, is the *curvature operator*.

Though it may appear to be a second order operator, the curvature operator is in fact linear: $\Omega(fX) \;=\; f\Omega(X)$ for all complex-valued functions f. Therefore we can express Ω as a matrix of 2-forms

$$\Omega\left(\frac{\partial}{\partial z^j}\right) \;=\; \Omega^i_j \otimes \frac{\partial}{\partial z^j}, \quad \Omega^i_j \;\in\; \bigwedge{}^{1,1} \tag{71}$$

Certainly $\Omega^i_j \in \bigwedge^2_{\mathbb{C}}$; that $\Omega^i_j \;\in\; \bigwedge^{1,1}$ is a consequence of compatibility with $\bar{\partial}$.

Theorem. The connection symbols Ω^i_j for any Chern connection are anti-Hermitian, in the sense that $\overline{\Omega^i_j} = -\Omega^j_i$. Further, $\Omega^i_j \in \bigwedge^{1,1}$, and we have

$$\Omega^i_j \;=\; \bar{\partial}\left(h^{i\bar{k}} \partial h_{j\bar{k}}\right). \tag{72}$$

One is struck by the comparative simplicity of this formula for curvature. Also,

$$\Omega^i_j \;=\; d\theta^i_j \,+\, \theta^i_t \wedge \theta^t_j. \tag{73}$$

Definition. The Chern curvature tensor is $\mathrm{Rm} = \sqrt{-1}\Omega^i_j dz^j \otimes \frac{\partial}{\partial z^i}$.

Now we are ready to relate the Chern connection to the usual connection.

Theorem. If the metric is Kähler so $d\omega = 0$, then the Chen connection is the Levi-Civita connection and the Chern curvature tensor is the Riemann curvature tensor. If the manifold is not Kähler, then the Chern connection has torsion, and the two curvature tensors agree on second order terms but not first order terms.

Due to Hermiticity, we can express the Chern curvature as

$$\mathrm{Rm} \;=\; \mathrm{Rm}_{i\bar{j}k}{}^l dz^i \otimes d\bar{z}^j \otimes dz^k \otimes \frac{\partial}{\partial z^l}. \tag{74}$$

and of course $\mathrm{Rm}_{i\bar{j}k\bar{l}} = \mathrm{Rm}_{i\bar{j}k}{}^s h_{s\bar{l}}$. The Chern curvature obeys $\overline{\mathrm{Rm}_{i\bar{j}k\bar{l}}} = \mathrm{Rm}_{j\bar{i}l\bar{k}}$ but has no other index symmetries unless the manifold is Kähler. The Second Bianchi identity always holds however; it's expression is $\nabla\Omega = 0$.

4.6 Curvature Computations

Given a Chern connection, we have three distinct Ricci curvature tensors

$$\mathrm{Ric}^{H\prime}{}_{i\bar{j}} \;=\; \mathrm{Rm}_{i\bar{j}k\bar{l}}h^{k\bar{l}}, \quad \mathrm{Ric}^{H\prime\prime}{}_{i\bar{j}} \;=\; \mathrm{Rm}_{i\bar{j}k\bar{l}}h^{i\bar{l}}, \quad \mathrm{Ric}^{H\prime\prime\prime}{}_{i\bar{j}} \;=\; \mathrm{Rm}_{i\bar{j}k\bar{l}}h^{k\bar{j}}. \tag{75}$$

It may grate upon real geometers to see an expression like $\mathrm{Rm}_{i\bar{j}k\bar{l}}h^{k\bar{l}}$, but in the complex setting this is not necessarily zero, which is a major difference between anti-Hermiticity and anti-symmetry. Each Ricci tensor is Hermitian:

$$\overline{\mathrm{Ric}^{H\prime}{}_{i\bar{j}}} = \mathrm{Ric}'_{j\bar{\imath}}, \quad \overline{\mathrm{Ric}^{H\prime\prime}{}_{i\bar{j}}} = \mathrm{Ric}''_{j\bar{\imath}}, \quad \overline{\mathrm{Ric}^{H\prime\prime\prime}{}_{i\bar{j}}} = \mathrm{Ric}'''_{j\bar{\imath}} \tag{76}$$

and therefore reach divides into a symmetric and antisymmetric part: by convention

$$\begin{aligned} \mathrm{Ric}^{H\prime} \;&=\; \frac{1}{2}\mathrm{Ric} + \frac{\sqrt{-1}}{2}\rho \\ \mathrm{Ric} = Sym\,\mathrm{Ric}^{H\prime} \;&=\; \mathrm{Ric}^{H\prime}{}_{i\bar{j}}\left(dz^i \otimes d\bar{z}^j + d\bar{z}^j \otimes dz^i\right) \\ \rho = 2\sqrt{-1}Alt\,\mathrm{Ric}^{H\prime} \;&=\; \sqrt{-1}\,\mathrm{Ric}^{H\prime}{}_{i\bar{j}}\,dz^i \wedge d\bar{z}^j. \end{aligned} \tag{77}$$

The first of the Ricci tensors is easiest to deal with. Indeed

$$\rho' \;=\; 2\sqrt{-1}\partial\bar{\partial}\log\,det(h_{\alpha\bar{\beta}}) \tag{78}$$

and the corresponding scalar curvature is

$$s \;=\; h^{i\bar{j}}\,\mathrm{Ric}_{i\bar{j}} \;=\; -\triangle\log\,det(h_{\alpha\bar{\beta}}). \tag{79}$$

In general the three Ricci curvatures are distinct; the exception is the Kähler case when they are all identical. Indeed many compatibilities appear in the Kähler case:

Theorem. Assume (M^{2n}, J, ω) is Kähler. Then Rm and Ric are J-invariant:

$$\begin{aligned} \mathrm{Rm}\,(JX, JY, JZ, JW) \;&=\; \mathrm{Rm}(X,Y,Z,W), \\ \mathrm{Rm}\,(X, Y, JZ, JW) \;&=\; \mathrm{Rm}(X,Y,Z,W), \\ \mathrm{Ric}(JX,\ JY) \;&=\; \mathrm{Ric}(X,Y). \end{aligned} \tag{80}$$

Further, Ric', Ric'', Ric''' all coincide with the standard Ricci tensor Ric, and

$$\begin{aligned} \rho(X,Y) \;&=\; \mathrm{Ric}(JX,Y) \\ s \;&=\; \frac{1}{(n-1)!} * (\rho \wedge \omega \wedge \cdots \wedge \omega) \end{aligned} \tag{81}$$

(where there are $(n-1)$ many copies of ω, and $*$ is the Hodge-star). Also $\mathrm{Rm}_{i\bar{j}k\bar{l}} = \mathrm{Rm}_{k\bar{l}i\bar{j}}$ and the first Bianchi identity holds: $\mathrm{Rm}(X,Y,Z,W) = \mathrm{Rm}(Y,Z,Y,W) = \mathrm{Rm}(Z,X,Y,W) = 0$.

4.7 The Hodge-star and J

Definition. The complex Hodge duality operator $\bar{*}$ is

$$\bar{*}\eta \triangleq \overline{*\eta}, \quad \eta \in \bigwedge{}^{*,*}. \tag{82}$$

Clearly this is not a tensor. The artificial introduction of complex conjugation is necessary to unsure compatibility with the Hermitian metric. Indeed if $\eta, \mu \in \bigwedge^1_{\mathbb{C}}$ then $\langle \eta, \mu \rangle = h^{i\bar{j}}\mu_i\overline{\eta_j}$ is the same as $\bar{*}\,(\eta \wedge \bar{*}\mu)$ (the dualization $\langle \eta, \mu \rangle \triangleq \langle \eta^{\sharp}, \mu^{\sharp} \rangle$ causes $\langle \cdot, \cdot \rangle$ on covectors to be skew in the first position).

The two Hodge-star maps are compatible with the bidegree decomposition of the exterior algebra. Indeed we have isomorphisms, resp. anti-isomorphisms

$$* \; : \; \bigwedge{}^{p,q} \longrightarrow \bigwedge{}^{n-q,n-p}, \qquad \bar{*} \; : \; \bigwedge{}^{p,q} \longrightarrow \bigwedge{}^{n-p,n-q}. \tag{83}$$

On real k-forms, that is on $\bigwedge^*_{\mathbb{R}} \subset \bigwedge^{*,*}_{\mathbb{C}}$, clearly $* = \bar{*}$.

Theorem. On any Hermitian manifold, the Kähler form is always a $(1,1)$-form and is always real: $\overline{\omega} = \omega$. In other words, $\omega \in \bigwedge^{1,1} \cap \bigwedge^2_{\mathbb{R}}$.

Definition. On any Hermitian manifold we have the linear maps called the *Lefschetz operator* $L : \bigwedge^{p,q} \to \bigwedge^{p+1,q+1}$ and its dual $\Lambda : \bigwedge^{p,q} \to \bigwedge^{p-1,q-1}$:

$$L\eta = \omega \wedge \eta, \qquad \Lambda = \bar{*}\,(\omega \wedge \bar{*}\eta) = *\,(\omega \wedge *\eta)\,. \tag{84}$$

Now we consider the case of dimension 4. We have two decompositions:

$$\bigwedge{}^2_{\mathbb{C}} = \bigwedge{}^{0,2} \oplus \bigwedge{}^{1,1} \oplus \bigwedge{}^{2,0} \qquad \textit{and} \qquad \bigwedge{}^2_{\mathbb{C}} = \bigwedge{}^+_{\mathbb{C}} \oplus \bigwedge{}^-_{\mathbb{C}} \tag{85}$$

(where $\bigwedge^+_{\mathbb{C}} = \bigwedge^+ \otimes \mathbb{C}$, etc.). Note the spaces $\bigwedge^{0,2}, \bigwedge^{2,0}$ are each one-dimensional and $\bigwedge^{1,1}$ is four-dimensional. These decompositions are reasonably compatible:

Theorem. We have

$$\bigwedge{}^+_{\mathbb{C}} = \bigwedge{}^{0,2} \oplus span_{\mathbb{C}}\{\omega\} \oplus \bigwedge{}^{2,0}, \qquad \bigwedge{}^-_{\mathbb{C}} = Proj_{\perp\omega}\left\{\bigwedge{}^{1,1}\right\}. \tag{86}$$

Indeed this gives us a powerful way of viewing almost complex structures on 4-manifolds. Give a metric g, a compatible almost complex structure J is precisely the same as choosing a section $\omega \in \bigwedge^+_{\mathbb{R}}$ with $\langle \omega, \omega \rangle_A = 1$. Then $J^i_j = \omega_{js}g^{si}$ is a g-compatible almost complex structure.

A tougher question, not addressed here, is whether this can be chosen to be Kähler.

4.8 J, $\bar{*}$, and exterior derivatives

Definition. On a complex manifold, the *complementary exterior derivative* is

$$d^c \;=\; \sqrt{-1}\left(\partial \,-\, \bar\partial\right), \qquad d^c : \bigwedge\nolimits^k_{\mathbb{C}} \,\to\, \bigwedge\nolimits^{k+1}_{\mathbb{C}} \tag{87}$$

One computes $\overline{d^c} = d^c$ so d^c is real, and d^c obeys the Leibniz rule, so is a derivation. We have $\partial = \frac{1}{2}\left(d + \sqrt{-1}d^c\right)$, $\bar\partial = \frac{1}{2}\left(d - \sqrt{-1}d^c\right)$.

All of the exterior derivatives can be dualized, to obtain codifferentials:

$$\begin{aligned} &d^* \;=\; (-1)^{nk+n+1} * d*, \quad d^{c*} \;=\; (-1)^{nk+n+1} * d^c*, \quad d^*, d^{c*} : \bigwedge\nolimits^k_{\mathbb{C}} \to \bigwedge\nolimits^{k-1}_{\mathbb{C}} \\ &\bar\partial^* = (-1)^{n(p+q)+n+1}\bar{*}\partial\bar{*}, \quad \bigwedge\nolimits^{p,q} \to \bigwedge\nolimits^{p,q-1}, \\ &\partial^* = (-1)^{n(p+q)+n+1}\bar{*}\bar\partial\bar{*}, \quad \bigwedge\nolimits^{p,q} \to \bigwedge\nolimits^{p-1,q} \end{aligned} \tag{88}$$

and consequently four different Hodge Laplacians:

$$\triangle_d = [d, d^*], \quad \triangle_{d^c} = [d^c, d^c*], \quad \triangle_{\bar\partial} = [\bar\partial, \bar\partial^*], \quad \triangle_\partial = [\partial, \partial^*]. \tag{89}$$

Each of these respects the bidegree decomposition; each is a map $\bigwedge^{p,q} \to \bigwedge^{p,q}$. In general, all four are distinct from one another. As usual the Kähler case is special:

Theorem. If (M^{2n}, J, ω) is Kähler then

$$\triangle_d \;=\; \triangle_{d^c} \;=\; 2\triangle_{\bar\partial} \;=\; 2\triangle_\partial. \tag{90}$$

On functions we have

$$\bar\partial f \;=\; \frac{1}{2}\left(d + \sqrt{-1}Jd\right) f \qquad \textit{and} \qquad -\,2\sqrt{-1}\partial\bar\partial f \;=\; d(Jdf). \tag{91}$$

One computes $dJdf(X,Y) = Hess(X, JY) - Hess(Y, JX)$, so $dJdf(J\cdot,\cdot)$ is a kind of J-anti-invariant Hessian, which makes it remarkable that $dJdf$ is actually metric-independent.

Once again the Kähler case is special:
Theorem. (The Calabi Lemma) Assume (M^{2n}, J, ω) is Kähler manifold, and $\eta \in \bigwedge^{p,p}$ is exact: $\eta = d\gamma$ for some $\gamma \in \bigwedge^{p-1,p} \oplus \bigwedge^{p,p-1}$. Then actually

$$\eta \;=\; \sqrt{-1}\partial\bar\partial\mu, \tag{92}$$

for some $\mu \in \bigwedge^{p-1,p-1}$. If η is a real (p,p)-form, then μ is a real $(p-1, p-1)$ form.

In the special case that $\eta \in \bigwedge^{1,1}$ is exact, there is a <u>function</u> φ with $\eta = \sqrt{-1}\partial\bar\partial\varphi$ called the *potential* of η.

4.9 The Kähler relations

Definition. The *weight operator* is the linear extension of

$$H:\bigwedge{}^{*,*}\to\bigwedge{}^{*,*},\qquad H(\eta)=(n-p-q)\eta\quad\text{when}\quad\eta\in\bigwedge{}^{p,q}\tag{93}$$

Theorem. On Kähler manifolds, we have the *Kähler relations*

$$[L,\Lambda]=H,\quad [L,d]=\left[L,\bar\partial\right]=0,\quad \left[L,\bar\partial^*\right]=-\sqrt{-1}\partial.\tag{94}$$

Immediate consequences are $[\Lambda,\partial^*]=\sqrt{-1}\bar\partial$, etc. Any complex manifold (M^{2n},J) has it *Dolbeault cohomology classes* $H^{p,q}$.

Definition. The vector space $H^{p,q}(M^{2n})$ is the set of equivalence classes

$$[\eta]\;=\;\left\{\eta'\in\bigwedge{}^{p,q}\;\middle|\;\bar\partial\eta'=0,\text{ and }\eta-\eta'=\bar\partial\mu\text{ some }\mu\in\bigwedge{}^{p,q-1}\right\}.\tag{95}$$

This is directly analogous to the deRham cohomology spaces. The Dolbeault case, like the deRham case, has a cup product on classes given by the exterior product on forms and so the sum of Dolbeault spaces $H^{*,*}=\bigoplus_{p,q=0}^n H^{p,q}$ forms an algebra.

Theorem. (The Dolbeault theorem) If (M^{2n},J) is a compact complex manifold, then each Dolbeault class $[\eta]\in H^{p,q}$ is represented by a unique (p,q)-form $\eta'\in[\eta]$ with $\triangle_{\bar\partial}\eta'=0$.

Theorem. The Kähler relations imply $[\triangle_d,L]=0$ on any Kähler manifold.

Theorem. As a consequence, any Dolbeault class on a Kähler manifold is a deRham class: if $[\eta]\in H^{p,q}(M^{2n})$ then $[\eta]\in H^{p+q}_{dR}(M^{2n},\mathbb{C})$. Further, this is exhaustive:

$$H^k_{dR}(M^{2n},\mathbb{C})\;=\;H^{k,0}(M^{2n})\oplus H^{k-1,1}(M^{2n})\oplus\;\dots\;\oplus H^{0,k}(M^{2n}).\tag{96}$$

Theorem. If (M^{2n},J,ω) is Kähler, then if $[\eta]\in H^{p,q}$ is a Dolbeault class then $[L\eta]\in H^{p+1,q+1}$ and $[\Lambda\eta]\in H^{p-1,q-1}$ are also Dolbeault classes. Also the action of the Lie algebra $span_{\mathbb{C}}\{L,\Lambda,H\}$ on $H^{*,*}$ forms an irreducible $\mathfrak{so}(2)$ representation.

This $\mathfrak{so}(2)$ representation greatly structures $H^{*,*}$. One result is Lefschetz duality:

$$H^{p,q}(M^{2n})\;\approx\;H^{n-q,n-p}(M^{2n})\tag{97}$$

given by L^{n-p-q} when $p+q<n$ and Λ^{p+q-n} when $p+q>n$. Combined with *Serre duality* $H^{p,q}(M^{2n})\approx H^{q,p}(M^{2n})$ given by complex conjugation, we also have $H^{p,q}(M^{2n})\approx H^{n-p,n-q}(M^{2n})$. The collection $H^{*,*}$ with its attendant symmetries is called the *Hodge diamond*.

4.10 Complex dimension 2

Complex 2-manifolds are real 4-manifolds, and so have a special theory associated with them. The W^+ tensor on Kähler 4-manifolds is very special:

Theorem. (Derdzinki [2]) If (M^4, J, ω) is a Kähler 4-manifold, then $W^+ : \bigwedge^2 \to \bigwedge^2$ can be written

$$W^+ \;=\; \frac{R}{24}\left(3\omega \otimes \omega \;-\; 2Id_{\bigwedge^+}\right) \tag{98}$$

and $\frac{1}{4}|W^+|^2 = \frac{R}{24}$. The formulas from §3.9 thereby simplify:

$$\begin{aligned} |\operatorname{Rm}|^2 \;&=\; \frac{R}{3} \;+\; 2|\,\mathring{\operatorname{Ric}}\,|^2 \;+\; |W^-|^2 \\ \frac{1}{4} * (\Omega\wedge\Omega) \;&=\; \frac{R}{24} \;-\; \frac{1}{4}|W^-|^2 \\ * Pf(\Omega) \;&=\; \frac{R}{12} \;-\; \frac{1}{2}|\,\mathring{\operatorname{Ric}}\,|^2 \;+\; \frac{1}{4}|W^-|^2. \end{aligned} \tag{99}$$

In the complex setting the curvature 2-form Ω is not antisymmetric as in the real setting, but anti-Hermitian, and so $Tr\,\Omega = \Omega_s^s$ is a 2-form, non-zero in general. In fact $\rho = \sqrt{-1}Tr\,\Omega$ as noted in §4.6. It is easily checked that $d\rho = 0$, so ρ determines a deRham class: $[\rho] \in H^2_{dR}(M^{2n};\mathbb{R})$. For reasons we will not describe (but see [5]), although ρ varies with the metric, the class $[\rho]$ does not. The *first Chern class* of a Kähler manifold is defined to be $c_1(M^{2n}) = \frac{1}{2\pi}[\rho]$. Thus we have the two additional characteristic classes

$$\begin{aligned} c_1 \;&=\; \frac{1}{2\pi}[\rho] \quad\in\quad H^2_{dR}(M^4,\mathbb{Z}) \\ c_1 \smile c_1 \;&=\; \frac{1}{4\pi^2}[\rho\wedge\rho] \quad\in\quad H^4_{dR}(M^4,\mathbb{Z}) \end{aligned} \tag{100}$$

(proving these are in $H^*_{dR}(M^4,\mathbb{Z})$ and not $H^*_{dR}(M^4,\mathbb{R})$ is a matter of some depth). Using $\rho = \frac{R}{4}\omega + \rho_-$ where ρ_- is the projection of ρ onto $\bigwedge^-$, we compute

$$c_1 \smile c_1 \;=\; \frac{1}{4\pi^2}\left[\left(\frac{1}{8}R^2 \;-\; \frac{1}{2}|\,\mathring{\operatorname{Ric}}\,|^2\right) dVol\right] \tag{101}$$

Note that by the decoposition (86), since $\rho \in \bigwedge^{1,1}$, its decomposition into $\bigwedge^+$ and $\bigwedge^-$ is precisely its decomposition into a piece parallel to ω and its part ρ_- perpendicular to ω, which then must have the property that $\rho_-(\cdot, J\cdot) = \mathring{\operatorname{Ric}}$.

5 Exercises

1. Verify directly that $dVol$ on 2-manifolds gives the parallelogram volume of the span of two vectors.

2. Verify abstractly that $dVol$ gives the parallelotope volume of n many vectors at a point of an n-manifold.

3. Show that $\frac{1}{2}(g \owedge g)$ is the identity $Id : \bigwedge^2 \to \bigwedge^2$, with the multiplication conventions of (16).

4. Verify the $A \owedge B$ obeys the usual symmetries of the Riemann tensor, including the first Bianchi identity, provided that A, B are symmetric. Show by example that none of these identities need hold if either A or B is not symmetric.

5. Verify that $|g \owedge g|^2 = 8n(n-1)$.

6. If the 2-tensors A and B are trace free and orthogonal, we have seen that $A \owedge g$ is always orthogonal to $B \owedge g$ What is the best way to generalize this?

7. Verify the norm-square decomposition of the Riemann tensor:

$$|\,\mathrm{Rm}\,|^2 \;=\; \frac{2}{n(n-1)}R^2 \;+\; \frac{4}{n-2}|\,\mathring{\mathrm{Ric}}\,|^2 \;+\; |W|^2. \tag{102}$$

8. In dimension 2, prove that $(*^{(1)})^2 = -Id$ on covectors (meaning $*^{(1)}$ is a kind of imaginary unit).

9. In dimension 4, prove that W^+ and W^- are orthogonal to one another.

10. Using the conformal invariance of W, prove that $g^{ij}B_{ij} = 0$ and that B is conformally invariant.

11. (Harder) Using the diffeomorphism invariance of Rm (and therefore of W), show that $B^s{}_{j,s} = 0$.

12. Verify that if A, B are both antisymmetric k-tensors, then $\langle A, B\rangle_A = *^{(n)}\left(A \wedge *^{(k)}B\right)$.

13. Verify that the exterior derivative d from (24) obeys axioms (i)-(iv).

14. Verify that any exterior operator d satisfying (i)-(iv) of §3.2 satisfies (24).

15. Verify, from (i)-(iv) of §3.2, that $dd\eta = 0$ for any $\eta \in \bigwedge^k$.

16. Verify the derivation bracket formulas (37).

17. Using (40), verify (41).

18. In the theorem from §4.2, prove that (*v*) is equivalent to (*vi*). Show that either of these implies (*iv*).

19. In the theorem from §4.2, prove that (*i*) is equivalent to (*ii*).

20. In the theorem from §4.2, prove that (*iv*) is equivalent to (*iii*) and (*ii*).

21. If $A = (A^i_j)$ is a 4×4 antisymmetric matrix, prove that $Pf(A) = A^1_2 A^3_4 - A^1_3 A^2_4 + A^1_4 A^2_3$. It is also possible to show that $(Pf(A))^2 = \det(A)$.

22. Prove that (60) actually defines an Hermitian metric, under the conditions stated there.

23. Verify (61) and verify (70).

24. If ∇ is a Chern connection, use the Leibniz rule to prove the second Bianchi identity: $\nabla\Omega = 0$.

25. If g is a metric on a 4-manifold and having chosen some $\omega \in \bigwedge^+_{\mathbb{R}}$, prove that the tensor $J^i_j = \omega_{js} g^{si}$ had $J^2 = -|\omega|^2$. If ω has unit norm, show that J is a g-compatible almost complex structure.

26. Using the Kähler relations, prove that $[\triangle_d, L] = 0$ on a Kähler manifold.

(Updated Oct 2018)

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