Gravitationally Inspired Metrics

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1 Introduction

Many interesting Riemannian metrics were discovered in the course of the physics project known as Euclidean Quantum Gravity [9]. EQG stems from the notion that one can quantize gravity by applying the Feynman “path integral” formulation of quantum mechanics to the setting of geometry, where, in principal, space-time histories would superimpose and interact and a most probable space-time path be computed.

Even in its original context the Feynman path integral formalism is more difficult in the Schrödinger case, and elementary treatments typically first establish the theory for “imaginary time,” which produces solutions of the heat equation (or the field theoretic equivalent), and then uses analytic continuation to extend this into the “real time” direction, which produces solutions of the Schrödinger equation (or the field theoretic equivalent). Similarly, the dream of EQG is to develop a mathematically solid path integral formalism for metrics on Riemannian 4-manifolds, interpret this as an “imaginary time” theory, and then somehow rotate the formalism into the “real time” direction via some kind of analytic continuation (or a Wick rotation into the Lorentzian metric signature), and thereby establish, hopefully, a well-defined theory of quantum gravity.

Sadly this project encountered too many difficulties and claimed too few successes to be considered a promising road to quantum gravity, and is now relatively active. One positive outcome, however, was the discovery of a great many “Euclidean gravitational instantons,” which are complete solutions to the vacuum field equations (that is, Ricci flat Riemannian manifolds)—these were thought to be important to the EQG because, by analogy with Yang-Mills gauge theories, presumably path integral amplitudes would maximize near such instantons.

Early researchers in EQG developed many instanton metrics. This started with the originating paper of Gibbon-Hawking [9] where the Euclidean Schwarzschild metric was written down. Other important early researcher was done by Page [21] [22], Gibbons-Pope [13], Gibbons-Perry [12] and of course Hawking [15] [16]. A significant culmination came in the development of the Gibbons-Hawking ansatz in [10] [11], where also the terminology “nuts” and “bolts” was systematized (first used informally by Page [21], a “nut” is a point-like zero of a killing field and a “bolt” is a zero-locus of a Killing field that is topologically a 2-sphere).

We present metrics that were considered important to EQG. The exception is the Taub-NUT family of metrics which are discussed in detail elsewhere in these notes.

2 Open Manifolds

2.1 Gödel Metric

The Lorentzian version is due to Kurt Gödel [14]. The Riemannian Gödel metric is

$$g \;=\; (dt + e^x dz)^2 \;+\; dx^2 \;+\; dy^2 \;+\; \frac{1}{2}e^{2x}dz^2. \tag{1}$$

The Lorentzian Gödel metric—with a negative on the first term—models a universe filled with a pressureless perfect fluid rotating about the y-axis, and has some peculiar physical properties, such as closed time-like curves.

Of course this has a covariant-constant Killing field $\frac{\partial}{\partial y}$, so this is a geometric cross product of a 3-dimensional space with $\mathbb{R}^1$. The metric is complete. An orthonormal frame is $\eta^1 = dt + e^x$, $\eta^2 = dx$, $\eta^3 = dy$,$\eta^4 = \frac{1}{\sqrt{2}}e^x dz$.

Curvature and Asymptotics

Scalar curvature is -1. The Ricci vector is

$$\mathcal{RV} \;=\; \left(\eta^1,\, -\eta^2,\, 0,\, -\eta^4\right) \tag{2}$$

and the Weyl tensors are

$$W^+ \;=\; -\frac{1}{6}\left(3\omega\otimes\omega - 2Id_{\bigwedge^+}\right), \quad W^- \;=\; -\frac{1}{6}\left(3\omega'\otimes\omega' - 2Id_{\bigwedge^-}\right) \tag{3}$$

where $\omega = \eta^2\wedge\eta^3 - \eta^1\wedge\eta^4$, $\omega' = \eta^2\wedge\eta^3 + \eta^1\wedge\eta^4$.

2.2 Euclidean Schwarzschild

The Euclidean Schwarzshild metric was first written down by Gibbons-Hawking [9] in support of their Euclidean quantum gravity project. It is the Wick rotation of the classic 1915 Schwarzschild metric. The Euclidean Schwarzschild metric is

$$g \;=\; \frac{1}{1-\frac{2m}{r}}dr^2+\left(1-\frac{2m}{r}\right)d\psi^2+r^2\left(d\theta^2+\sin^2\theta\,d\varphi^2\right) \tag{4}$$

for $r\in(2m,\infty)$, $\psi\in[0,8m\pi)$. There is a coordinate singularity at $r=2m$; to see its resolution, we focus on the fiber parameterized by r,ψ and substitute $\rho^2=8m(r-2m)$ to get

$$\begin{aligned} g_{fiber} \;&=\; \left(\frac{1}{16m^2}\rho^2+1\right)d\rho^2+\frac{\rho^2}{\rho^2+16m^2}d\psi^2 \\ &=\; \left(1+O(\rho^2)\right)d\rho^2+\frac{1}{16m^2}\left(\rho\,+\,O(\rho^3)\right)^2d\psi^2 \end{aligned} \tag{5}$$

Therefore as ρ nears 0, the metric looks like $d\rho^2+\rho^2 d\left(\psi/4m\right)^2$, which is smooth (not cone-like) provided the range of $\frac{\psi}{4m}$ is $[0,2\pi)$, or $\psi\in[0,8m\pi)$.

Topologically, this metric exists on $\mathbb{S}^2\times\mathbb{R}^2$, with a "zero section" at $r=2m$ where the metric has a bolt-like resolution: a 2-sphere of volume $2m$. Sometimes this "bolt" is called the Euclidean event horizon.

Euclidean Schwarzschild is never Kähler. (There is a different family of scalar-flat Kähler metrics on $\mathbb{S}^2\times\mathbb{R}^2$, discussed elsewhere).

Curvature and Asymptotics

This metric is Ricci-flat. Its Weyl curvatures are

$$W^+ \;=\; \frac{m}{2r^3}\left(3\omega\otimes\omega-2Id_{\bigwedge^+}\right),\quad W^- \;=\; \frac{m}{2r^3}\left(3\omega'\otimes\omega'-2Id_{\bigwedge^-}\right). \tag{6}$$

where $\omega=dr\wedge d\psi+r^2\sin\theta\,d\theta\wedge d\varphi$, $\omega'=dr\wedge dt-r^2\cos\theta\,d\theta\wedge d\varphi$

Asymptotically we have $g\approx dr^2+d\psi^2+r^2(d\theta^2+\sin^2\theta\,d\varphi^2)$ so r is roughly the distance function. Therefore $|\,\mathrm{Rm}\,|=O(r^{-3})$ and volumes increase like $O(r^3)$.

2.3 Euclidean Schwarzschild-anti deSitter

Schwarzshild-adS is the version of Schwarzschild with a negative Einstein constant.

$$g = \frac{1}{1-\frac{\Lambda}{3}r^2-\frac{2m}{r}}dr^2 + \left(1-\frac{\Lambda}{3}r^2-\frac{2m}{r}\right)d\psi^2 + r^2\left(d\theta^2+\sin^2\theta\,d\varphi^2\right). \tag{7}$$

for $r \in (r_1, r_2)$, $\psi \in [0, \psi_0)$. For $r_1 < r_2$ to be real, $-\infty < \Lambda < \frac{1}{9m^2}$; one can find r_1, r_2 explicitly by using the cubic equation. Of course r_2 is $+\infty$ in the anti deSitter case $\Lambda \leq 0$. When $\Lambda = 0$ this is the Euclidean Schwarzschild metric.

Curvature and asymptotics

This metric is Einstein with Einstein constant Λ, so $\mathring{\mathrm{Ric}} = 0$, and $R = 4\Lambda$.

The Weyl curvatures are

$$W^+ = \frac{m}{2r^3}\left(3\omega\otimes\omega - 2Id_{\bigwedge^+}\right), \quad W^- = \frac{m}{2r^3}\left(3\omega'\otimes\omega' - 2Id_{\bigwedge^-}\right) \tag{8}$$

where $\omega = dr\wedge dt + r^2\sin\theta d\theta\wedge d\varphi$, $\omega' = dr\wedge dt - r^2\sin\theta d\theta\wedge d\varphi$.

In the adS case, where $\Lambda < 0$, we have $r = Exp(dist)$. Balls expand exponentially and $W^{\pm}$ decays logarithmically.

2.4 Burns, Eguchi-Hanson, and Lebrun metrics on $O(-k)$

Eguchi-Hanson [8] and Calabi [2] independently discovered the Eguchi-Hanson metric on $O(-2)$, Burns [1] discovered a scalar-flat Kähler metric on $O(-1)$, and Lebrun [20] generalized these constructions to a family of scalar-flat Kähler metrics on $O(-k)$, all $k \in \{1, 2, \dots\}$. These can be expressed

$$g \;=\; \frac{1}{U}dr^2 + r^2\left(\eta_X^2 + \eta_Y^2\right) + r^2U\eta_Z^2, \quad U \;=\; \left(1 - \frac{m^2}{r^2}\right)\left(1 + \frac{(k-1)m^2}{r^2}\right). \tag{9}$$

To resolve the coordinates singularity at $r = m$, we use $r = \rho^2 + m$ and examine the metric on the 2-plane parameterized by r, ψ:

$$\begin{aligned} g_{fiber} \;&=\; \frac{(\rho^2+m)^4}{(\rho^2+2m)(\rho^4+2m\rho^2+km^2)}d\rho^2 + \frac{(\rho^2+2m)(\rho^4+2m\rho^2+km^2)}{4(\rho^2+m)^2}\rho^2d\psi^2 \\ &=\; \frac{2m}{k}\left[(1+O(\rho^2))d\rho^2 \;+\; (1+O(\rho^2))\rho^2\left(d\,\frac{k}{2}\psi\right)^2\right] \end{aligned} \tag{10}$$

and so we require $\psi \in [0, \frac{4\pi}{k})$ in order for this metric to be smooth (not conical) near $\rho = 0$. Thus we make the identification $\psi \approx \psi + \frac{4\pi}{k}$, which makes the level-sets $r = const$ into lens spaces $L(k, 1)$.

Topologically, the Lebrun metric with parameter $k \in \{1, 2, \dots\}$ is a metric on $O(-k)$. These metrics are always Kähler with Kähler form ω (see introduction).

Curvature and Asymptotics

For each $k = 1, 2, \dots$ these metrics are scalar-flat and half-conformally flat. They are Ricci-flat if and only if $k = 2$, which is the Eguchi-Hanson instanton. The Ricci vector is

$$\mathcal{RV} \;=\; \frac{2(k-2)m^2}{r^4}\left(\eta^1, -\eta^2, -\eta^3, \eta^4\right) \tag{11}$$

and the Weyl tensors are

$$W^+ \;=\; 0, \quad W^- \;=\; \frac{2(k-1)m^4 - (k-2)m^2r^2}{r^6}\left(3\omega' \otimes \omega' \;-\; Id_{\bigwedge^-}\right) \tag{12}$$

Curvature decay is $O(r^{-4})$—except for Eguchi-Hanson case $k = 2$ when decay is $O(r^{-6})$—and volume growth is $O(r^4)$. These are ALE (asymptotically locally Euclidean) for all $k \in \mathbb{N}$; the Burns metric $k = 1$ is AE (asymptotically Euclidean).

2.5 Euclidean Kerr

The Lorentzian Kerr metric [17] [18] was discovered using geometric algebra techniques to express certain metrics more simply. It models a rotating black hole in the center of empty space with no cosmological constant. The Euclidean Kerr metric is

$$g = \Sigma\left(\frac{1}{\triangle_r}dr^2 + d\theta^2\right) + \frac{1}{\Sigma}\left(\sin^2\theta\left(\alpha d\psi + P_r d\varphi\right)^2 + \triangle_r\left(d\psi + P_\theta\, d\varphi\right)^2\right) \tag{13}$$

where

$$P_r = r^2 - \alpha^2, \quad P_\theta = -\alpha\sin^2(\theta), \quad \triangle_r = r^2 - 2mr - \alpha^2, \quad \Sigma = r^2 - \alpha^2\cos^2\theta. \tag{14}$$

This metric appears in [7], although there is a slight typo there. See also [5]. The parameter m is the "mass" and α is known as the rotation density. We must have $\alpha \in [0,1)$; the upper bound on α means physically that a black hole of a given mass has a maximal angular momentum. The coordinate ψ is the Wick-rotated time coordinate. When $\alpha = 0$ this is the Schwarzschild metric.

Topologically, this is a smooth metric on $\mathbb{S}^2 \times \mathbb{R}^2$ assuming two quantizations:

$$\psi \;\in\; \left[0,\, 2\pi\frac{1}{\kappa}\right), \quad \varphi \;\in\; \left[0,\, 2\pi\frac{\alpha}{\sqrt{m^2+\alpha^2}}\right), \tag{15}$$

where $\kappa = \frac{\sqrt{m^2+\alpha^2}}{2m(m+\sqrt{m^2+\alpha^2})}$, and the bolt occurs at $r = m + \sqrt{m^2+\alpha^2}$. These metrics are Ricci-flat, and are never Kähler.

Curvature and Asymptotics

Euclidean Kerr is a family of Ricci-flat metrics on $\mathbb{S}^2 \times \mathbb{R}^2$ parameterized by α and m. The Weyl tensors are

$$\begin{aligned} W^+ &= \frac{m/2}{(r-\alpha\cos\theta)^3}\left(3\omega\otimes\omega - 2Id_{\bigwedge^+}\right), \quad \text{and} \\ W^- &= \frac{m/2}{(r+\alpha\cos\theta)^3}\left(3\omega'\otimes\omega' - 2Id_{\bigwedge^-}\right). \end{aligned} \tag{16}$$

Curvature falls off cubically.

2.6 Kerr-Newmann, *a.k.a.* Kerr-Schild

The Lorentzian metrics known as the Kerr-Newmann or the Kerr-Schild metrics model a charged, rotating black hole. Its Wick rotation is the Euclidean Kerr-Newmann (aka Kerr-Schild) metric:

$$g = \Sigma\left(\frac{1}{\triangle_r}dr^2 + d\theta^2\right) + \frac{1}{\Sigma}\left(\sin^2\theta\,(\alpha d\psi + P_r d\varphi)^2 + \triangle_r\,(d\psi + P_\theta\, d\varphi)^2\right) \tag{17}$$

where

$$P_r = r^2 - \alpha^2,\ P_\theta = -\alpha\sin^2(\theta),\ \Sigma = r^2 - \alpha^2\cos^2\theta,\ \triangle_r = r^2 - 2mr - \alpha^2 - Q^2. \tag{18}$$

The parameter m is the mass, α is the black hole's rotation density, and Q is the black hole's *effective charge*: $Q^2 = e^2 - p^2$ where e is its electrical charge and p is its magnetic charge. Because charges create Maxwell fields and therfore non-zero stress-energy tensor, these metrics are not Ricci-flat unless $Q = 0$. Kerr-Newmann satisfies the field equations

$$\mathrm{Ric}_{ij} - \frac{R}{2}g_{ij} \;=\; 2\left(F_{is}F_j{}^s - \frac{1}{4}|F|^2 g_{ij}\right) \tag{19}$$

where the Maxwell tensor $F = dA$ is determined by the potential

$$A = -\frac{p\cos\theta}{\Sigma}\left(\alpha d\psi + P_r d\varphi\right) - \frac{e\,r}{\Sigma}\left(d\psi + P_\theta d\varphi\right). \tag{20}$$

Topologically, these metrics exist on $\mathbb{S}^2 \times \mathbb{R}^2$; to see this one must find the correct quantizations on ψ, φ, which is an involved task but was carried out in [5]. These metrics are scalar flat, and never Kähler.

Curvature and Asymptotics

Kerr-Newmann is scalar-flat. Its Ricci vector is

$$\mathcal{RV} \;=\; \frac{Q^2}{(r^2 - \alpha^2\cos^2(\theta))^2}\left(\eta^1, -\eta^2, -\eta^3, \eta^4\right) \tag{21}$$

and its Weyl tensors are

$$\begin{aligned} W^+ &= \frac{m(r + \alpha\cos\theta) + Q^2}{2(r - \alpha\cos\theta)^3(r + \alpha\cos\theta)}\left(3\omega\otimes\omega - 2Id_{\bigwedge^+}\right), \quad \text{and} \\ W^- &= \frac{m(r - \alpha\cos\theta) + Q^2}{2(r - \alpha\cos\theta)(r + \alpha\cos\theta)^3}\left(3\omega'\otimes\omega' - 2Id_{\bigwedge^-}\right). \end{aligned} \tag{22}$$

We observe quartic Ricci decay and cubic Weyl curvature decay. Volume growth is cubic.

2.7 Kerr-Newmann-anti deSitter

The widest known explicit generlization of the Kerr metric is the Kerr-Newmann metric with cosmological constant. The Lorentzian version was written down by Carter [4] and Plebański-Demiański [24]. The Euclidean version [5] is

$$\begin{aligned} g \;=\; \Sigma\left(\frac{1}{\triangle_r}dr^2+\frac{1}{\triangle_\theta}d\theta^2\right) \\ +\frac{1}{\Xi^2\Sigma}\left(\triangle_\theta\sin^2\theta\left(\alpha d\psi+P_r d\varphi\right)^2+\triangle_r\left(d\psi+P_\theta\,d\varphi\right)^2\right) \end{aligned} \tag{23}$$

where

$$\begin{aligned} &P_r=r^2-\alpha^2, \quad P_\theta=-\alpha\sin^2(\theta), \quad \Sigma=r^2-\alpha^2\cos^2\theta, \quad \Xi=1-\Lambda\alpha^2 \\ &\triangle_r=(r^2-\alpha^2)(1-\Lambda r^2)-2mr-Q^2, \quad \triangle_\theta=1-\Lambda\alpha^2\cos\theta. \end{aligned} \tag{24}$$

The parameter m is mass, $\alpha\in[0,1)$ is the rotational density, Q is called the effective charge, and Λ is the cosmological constant. The Kerr-Newmann metric is $\Lambda=0$.

Curvature and Asymptotics

Kerr-Newmann has constant scalar curvature. Scalar curvature is 12Λ and the Ricci vector is

$$\mathcal{RV} \;=\; 3\Lambda\left(\eta^1,\eta^2,\eta^3,\eta^4\right)+\frac{Q^2}{(r^2-\alpha^2\cos^2(\theta))^2}\left(\eta^1,-\eta^2,-\eta^3,\eta^4\right) \tag{25}$$

and its Weyl tensors are

$$\begin{aligned} W^+ &= \frac{m(r+\alpha\cos\theta)+Q^2}{2(r-\alpha\cos\theta)^3(r+\alpha\cos\theta)}\left(3\omega\otimes\omega-2Id_{\bigwedge^+}\right), \quad \text{and} \\ W^- &= \frac{m(r-\alpha\cos\theta)+Q^2}{2(r-\alpha\cos\theta)(r+\alpha\cos\theta)^3}\left(3\omega'\otimes\omega'-2Id_{\bigwedge^-}\right). \end{aligned} \tag{26}$$

In the anti deSitter case $\Lambda<0$, volume growth is exponential. We observe logarithmic Weyl curvature decay, and of course no Ricci curvature decay.

The deSitter case $\Lambda>0$, discussed in §3.6 and analyzed in [5], leads to compact manifolds provided certain quantization conditions on the parameters are met.

2.8 Rotating Taub-Bolt

Discovered by Gibbons and Perry [12], this metric is

$$
\begin{aligned}
g \;&=\; \Sigma\left(\frac{1}{\triangle}dr^2 + d\theta^2\right) + \frac{1}{\Sigma}\left(\sin^2\theta\,(\alpha d\psi + P_r d\varphi)^2 + \triangle\,(d\psi + P_\theta d\varphi)^2\right), \quad \text{where} \\
P_r &= r^2 - \alpha^2 - n^2 - \frac{\alpha^2 n^2}{n^2-\alpha^2} \\
P_\theta &= -\alpha\sin^2\theta + 2n\cos\theta - \frac{\alpha n^2}{n^2-\alpha^2} \\
\Sigma \;&=\; r^2 - (n+\alpha\cos\theta)^2 \\
\triangle \;&=\; r^2 - 2mr + n^2 - \alpha^2.
\end{aligned}
\tag{27}
$$

Surprisingly, this Ricci-flat metric has never been satisfactorally analyzed. A Misner string (codimension-2 coordinate singularity) and two Dirac strings (codimension-3 coordinate singularity) must be resolved, which will impose quantization conditions on the parameters m, n and α and on one or more of the coordinates.

If $\alpha = 0$ this is the extended Taub-NUT metric. If also $m = 2.5n$, this is Taub-bolt.

This metric is sometimes called the Kerr-NUT metric.

Topologically, the nature of this metric is not yet known. Conceivably there are choices of parameters and consequent coordinate quantizations both with nut-style completions and with bolt-style completions.

Curvature and Asymptotics

The rotating Taub-bolt metric is Ricci flat. Its Weyl tensors are

$$
\begin{aligned}
W^+ \;&=\; \frac{m-n}{2(r-n-\alpha\cos\theta)^3}\left(3\omega\otimes\omega - 2Id_{\bigwedge^+}\right) \\
W^- \;&=\; \frac{m+n}{2(r+n+\alpha\cos\theta)^3}\left(3\omega'\otimes\omega' - 2Id_{\bigwedge^-}\right)
\end{aligned}
\tag{28}
$$

Curvature decays cubically.

3 Compact Manifolds

3.1 $\mathbb{S}^2 \times \mathbb{S}^2$

Among the many ways of expressing this metric, we write

$$g \;=\; \frac{1}{1-\Lambda r^2}dr^2 + (1-\Lambda r^2)d\tau^2 \;+\; \frac{1}{\Lambda}\left(d\theta^2 \;+\; \sin^2\theta d\varphi^2\right) \tag{29}$$

taken from [7]. This metric is Einstein with Einstein constant Λ and scalar curvature Λ. The Weyl curvatures are

$$W^+ \;=\; \frac{\Lambda}{6}\left(3\omega\otimes\omega - 2Id_{\Lambda^+}\right), \quad W^- \;=\; \frac{\Lambda}{6}\left(3\omega'\otimes\omega' - 2Id_{\Lambda^-}\right) \tag{30}$$

where $\omega = dr\wedge d\tau + \Lambda\sin\theta\, d\theta\wedge d\varphi$, $\omega' = dr\wedge d\tau - \Lambda\sin\theta\, d\theta\wedge d\varphi$. This metric is Kähler with Kähler form ω.

3.2 $\mathbb{S}^4$

Among the many ways of expressing this metric, we may write

$$\begin{aligned} g &= dr^2 + \Lambda^2 \sin^2(r/\Lambda)\left(\eta_X^2 + \eta_Y^2 + \eta_Z^2\right), \quad \textit{or} \\ g &= \frac{4}{(1+\Lambda r^2)^2}\left(dr^2 + \frac{1}{4}\left(d\theta^2 + \sin^2\theta d\varphi^2\right) + \frac{1}{4}\left(d\psi + \cos\theta\, d\varphi\right)^2\right) \end{aligned} \tag{31}$$

This metric is Einstein with Einstein constant 3Λ and scalar curvature 12Λ. The Weyl curvatures are zero (the metric is conformally flat). This metric is not Kähler.

3.3 $\mathbb{C}P^2$

We write the metric in two ways:

$$
\begin{aligned}
g &= \frac{1}{\left(1+\frac{\Lambda}{6}r^2\right)^2}\left(dr^2+r^2\eta_X^2\right)+\frac{r^2}{1+\frac{\Lambda}{6}r^2}\left(\eta_Y^2+\eta_Z^2\right), \quad r\in[0,\infty) \\
g &= \frac{3}{2\Lambda}\left[dr^2\;+\;4\sin^2(r/2)\left(\eta_X^2+\eta_Y^2\right)\;+\;\sin^2(r)\eta_Z^2\right], \quad r\in(0,\pi).
\end{aligned}
\tag{32}
$$

These are both a multiple of the Fubini-Study metric, described elsewhere. This metric can be considered a limit of the Burns metric with positive cosmological constant.

This metrics are Einstein with Einstein consant Λ, and are half-conformally flat:

$$
W^+ \;=\; \frac{\Lambda}{6}\left(3\omega\otimes\omega-2Id_{\Lambda^+}\right), \quad W^- \;=\; 0 \tag{33}
$$

where ω is the Kähler form.

3.4 Page metric on $\mathbb{C}P^2\sharp\overline{\mathbb{C}P}^2$

The Page metric [21] on $\mathbb{C}P^2\sharp\overline{\mathbb{C}P}^2$ is

$$\begin{aligned} g &= U^{-1}dr^2 + 4\frac{1-\nu^2\cos^2(r)}{3+6\nu^2-\nu^4}\left({\eta_X}^2+{\eta_Y}^2\right) + \frac{\sin^2(r)}{(3+\nu^2)^2}U{\eta_Z}^2, \\ U &= \frac{3-\nu^2-\nu^2(1+\nu^2)\cos^2(r)}{1-\nu^2\cos^2(r)} \end{aligned} \tag{34}$$

This metric is Einstein if and only if ν is the positive root of $\nu^4+4\nu^3-6\nu^2+12\nu-3=0$; this choice of ν gives the *Page metric*. One notices that this metric has nearly, but not exactly, the form of an extended Taub-NUT metric with cosmological constant.

This metric was obtained by studying a Taub-bolt style metric with positive cosmological constant. The positive cosmological constant forces the fibers of the $\mathcal{O}(-1)$ bundle to curve back down, where a new bolt is required to complete the metric. The new bolt forces a second set of quantization conditions, which forces a value on the cosmological constant (through the parameter ν).

However, as Page found, both quantization conditions cannot simultaneously be met using the Taub-NUT rubric. Page's move was to take a limit as the two conditions get closer and closer to being met—called the Page limit—and showed through a coordinate transformation that the resulting object is indeed smooth.

Curvature and Asymptotics

With the correct choice of ν (which is about $\nu = 0.2817$). Scalar curvature is $R = \frac{837+1773\nu^2+762\nu^4+114\nu^6-31\nu^8+\nu^{10}}{8(3+\nu^2)^2}$, which is about 12.9523. The metric is Einstein if and only if ν has the correct value, in which case the Ricci vector is

$$\mathcal{RV} = \frac{R}{4}\left(\eta^1,\eta^2,\eta^3,\eta^4\right). \tag{35}$$

The Weyl tensors are

$$W^+ = F(r)\left(3\omega\otimes\omega - Id_{\bigwedge^+}\right), \quad W^- = G(r)\left(3\omega'\otimes\omega' - Id_{\bigwedge^-}\right) \tag{36}$$

where $F(r)$ and $G(r)$ are certain rational expressions in $\cos(r)$ and $\sin(r)$. Roughly $F(r) = \frac{-949.0-730.1\cos(r)-100.9\cos(2r)-4.7\cos(3r)}{(12.6-\cos^2(r))^3}$, $G(r) = \frac{-949.0+730.1\cos(r)-100.9\cos(2r)+4.7\cos(3r)}{(12.6-\cos^2(r))^3}$.

This metric is not Kähler, but it is conformal to a Kähler metric with conformal factor the inverse of the coefficeient on W^+; this Kähler metric is one of the extremal Kähler metrics discovered by Calabi [3] and described elsewhere in these notes.

3.5 Euclidean Schwarzschild-deSitter

Schwarzshild-deSitter is the version of Schwarzschild with positive Einstein constant:

$$g = \frac{1}{1-\frac{\Lambda}{3}r^2-\frac{2m}{r}}dr^2 + \left(1-\frac{\Lambda}{3}r^2-\frac{2m}{r}\right)d\psi^2 + r^2\left(d\theta^2+\sin^2\theta\, d\varphi^2\right) \tag{37}$$

for $r \in (r_1, r_2)$, $\psi \in [0, \psi_0)$. For $r_1 < r_2$ to be real, $-\infty < \Lambda < \frac{1}{9m^2}$; one can find r_1, r_2 explicitly by using the cubic equation. Of course r_2 is $+\infty$ in the anti deSitter $\Lambda < 0$. The $\Lambda = 0$ case is the standard Schwarzschild metric.

In the dS case where $0 < \Lambda < \frac{1}{9m^2}$, the metric closes off at both $r = r_1$ and at $r = r_2$, so we require a bolt at both ends. If the metric were to be smooth, we would require two quantization conditions for the coordinate ψ. However one can show that this can occur if and only if the roots coincide: $r_1 = r_2$, in which case the metric is clearly singular.

Nevertheless, sending $\Lambda \nearrow \frac{1}{9m^2}$ and sending $r_2 - r_1 \searrow 0$, we can take a Gromov-Hausdorff limit, which gives the standard metric on the product $\mathbb{S}^2 \times \mathbb{S}^2$ [13], which we may regard as the compact Schwarzschild metric. This limiting method, known as the *Page limit*, was also described briefly in [11].

Topologically, for $\Lambda \in (0, \frac{1}{9m^2})$ each metric (37) is a conifold metric on $\mathbb{S}^2 \times \mathbb{S}^2$. The metric is <u>never smooth</u> for positive Λ; there is always a conical singularity along at least one of the bolts. The Page limit, which *is* smooth, can be called the Euclidean Schwarzschild-deSitter metric on $\mathbb{S}^2 \times \mathbb{S}^2$, is in fact the product metric.

Curvature and asymptotics

This metric is Einstein with Einstein constant Λ, so $\mathring{\mathrm{Ric}} = 0$, and $R = 4\Lambda$.

The Weyl curvatures of (37) are

$$W^+ = \frac{m}{2r^3}\left(3\omega\otimes\omega - 2Id_{\bigwedge^+}\right), \quad W^- = \frac{m}{2r^3}\left(3\omega'\otimes\omega' - 2Id_{\bigwedge^-}\right) \tag{38}$$

where $\omega = dr \wedge dt + r^2 \sin\theta d\theta \wedge d\varphi$, $\omega' = dr \wedge dt - r^2 \sin\theta d\theta \wedge d\varphi$. However in the Page limit where we obtain $\mathbb{S}^2 \times \mathbb{S}^2$, the Weyl tensors are constant, as in (30).

In the deSitter case, where $\Lambda > 0$, the object is always compact after attaching appropriate bolts. But the metric is always a conifold, not a manifold.

3.6 Euclidean Kerr-Newmann-deSitter

The Euclidean Kerr-Newmann-deSitter metric is

$$g = \Sigma\left(\frac{1}{\triangle_r}dr^2 + \frac{1}{\triangle_\theta}d\theta^2\right) + \frac{1}{\Xi^2\Sigma}\left(\triangle_\theta \sin^2\theta\,(\alpha d\psi + P_r d\varphi)^2 + \triangle_r\,(d\psi + P_\theta\, d\varphi)^2\right) \tag{39}$$

where

$$\begin{aligned} &P_r = r^2 - \alpha^2, \quad P_\theta = -\alpha\sin^2(\theta), \quad \Sigma = r^2 - \alpha^2\cos^2\theta, \quad \Xi = 1 - \Lambda\alpha^2 \\ &\triangle_r = (r^2-\alpha^2)(1-\Lambda r^2) - 2mr - Q^2, \quad \triangle_\theta = 1 - \Lambda\alpha^2\cos\theta. \end{aligned} \tag{40}$$

(This is identical to (23).) The parameter m is mass, $\alpha \in [0,1)$ is rotational density, Q is effective charge, and Λ is the cosmological constant. The deSitter case is that $\Lambda > 0$, in which case the metric is singular for two values of r, and so two bolts (or, conceivably, nuts) must be attached to complete the metric. Many appropriate quantization conditions were worked out in [5].

Curvature and Asymptotics

This metric has constant scalar curvature, and is Einstein if and only if $Q = 0$. Scalar curvature is 12Λ and the Ricci vector is

$$\mathcal{RV} \;=\; 3\Lambda\left(\eta^1, \eta^2, \eta^3, \eta^4\right) + \frac{Q^2}{(r^2 - \alpha^2\cos^2(\theta))^2}\left(\eta^1, -\eta^2, -\eta^3, \eta^4\right) \tag{41}$$

and the Weyl tensors are

$$\begin{aligned} W^+ &= \frac{m(r+\alpha\cos\theta) + Q^2}{2(r-\alpha\cos\theta)^3(r+\alpha\cos\theta)}\left(3\omega\otimes\omega - 2Id_{\bigwedge^+}\right), \quad \text{and} \\ W^- &= \frac{m(r-\alpha\cos\theta) + Q^2}{2(r-\alpha\cos\theta)(r+\alpha\cos\theta)^3}\left(3\omega'\otimes\omega' - 2Id_{\bigwedge^-}\right). \end{aligned} \tag{42}$$

The deSitter case $\Lambda > 0$ was analyzed by [5], and leads to compact manifolds when certain quantization conditions on the parameters are met. The quantization conditions are of sufficient complexity that we do not discuss them; see [5].

The authors of [5] obtained quantization conditions that guarantee the object is a manifold, but unfortunately do not fully characterize the topological types obtained. The limiting case of one of these should be the Page metric.

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