Metrics on bundles over spheres.

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1 Introduction

We examine various metrics on 2-plane bundles over $\mathbb{S}^2$ that can be written with the squashed-spheres rubric or as $\mathbb{S}^2 \times \mathbb{R}^2$, but which don't fit the Taub-NUT style rubric. The base spaces are twisted 2-plane bundles over the sphere. In algebraic geometry terms, the underlying topological spaces are $\mathcal{O}(-k)$ for $k \in \{0, 1, 2, \dots\}$. In terms of differential geometry, these are sometimes called the Dirac monopoles. Recall that the total space of the cotangent bundle $T^*\mathbb{S}^2$ is $\mathcal{O}(-2)$.

The study of such metrics, and many others, seems to begin with the Gibbons-Hawking paper [5], where they wrote down the Euclidean Schwarzschild metric; their idea was the "Wick rotation." Although more an intuitive rule of thumb than a rigorous general construction, the Wick rotation is the replacement of the Lorenzian time coordinate t with $\sqrt{-1}t$, thereby "rotating" time to "imaginary time." Becuase $-dt^2$ then becomes dt^2, the effect is to turn a metric of signature $(-,+,+,+)$ to a metric of signature $(+,+,+,+)$.

This is certainly not a rigorous process. The conceit is that the coordinates t, x, y, z each exist on the complex plane so a larger 4-complex dimensional, 8-real dimensional manifold exists; the Lorentzian and Euclidean version of the metric correspond to choosing different "real slices" of this larger object.

Most of the resulting "imaginary time" metrics (that is, Riemannian metrics) require special considerations to make them smooth. Often "mass" parameters must also be rotated: m becomes $\sqrt{-1}m$, but more seriously the resulting Euclidean metrics often become singular at certain locations, and there does not seem to be a general way of resolving such singularities—although a fairly broad rule is that event horizons in the Lorentzian setting become "nuts" (points) or "bolts" (2-spheres) in the Riemannian setting. Most metrics require some special process, adapted to the specific case under consideration, for resolving singularities. They usually require some quantization condition on its parameters or coordinates that are absent in the Lorentzian cases.

Below we make use of 2-forms $\omega \in \bigwedge^+$ and $\omega' \in \bigwedge^-$. If the metric is $g = Adr^2 + B\eta_X^2 + C\eta_Y^2 + D\eta_Z^2$, then we are using $\omega = \sqrt{AC}dr \wedge \eta_Y + \sqrt{BD}\eta_X \wedge \eta_Z$ and $\omega' = \sqrt{AC}dr \wedge \eta_Y - \sqrt{BD}\eta_X \wedge \eta_Z$.

2 Metrics on bundles over $\mathbb{S}^2$

2.1 $\mathbb{S}^2 \times \mathcal{H}^2$

This is the sphere crossed with hyperbolic spaces. Among the many ways of expressing these metric, we choose:

$$g \;=\; \frac{1}{1+\Lambda^1 r^2} dr^2 + (1+\Lambda^1 r^2) d\tau^2 \;+\; \frac{1}{\Lambda^2}\left(d\theta^2 \;+\; \sin^2\theta d\varphi^2\right) \tag{1}$$

where τ is a cyclic variable. This metric has scalar curvature $R = 2(\lambda^2 - \lambda^1)$. The Ricci vector is

$$\mathcal{RV} \;=\; \left(-\Lambda^1\eta^1, -\Lambda^1\eta^2, \Lambda^2\eta^3, \Lambda^2\eta^4\right) \tag{2}$$

where $\eta^1, \eta^2, \eta^3, \eta^4$ are the natural unit covectors. The Weyl tensors are

$$W^+ \;=\; \frac{\Lambda^2 - \Lambda^1}{12}\left(3\omega \otimes \omega - 2Id_{\Lambda^+}\right), \quad W^- \;=\; \frac{\Lambda^2 - \Lambda^1}{12}\left(3\omega' \otimes \omega' - 2Id_{\Lambda^-}\right) \tag{3}$$

When $\Lambda^1 = \Lambda^2$, the metric is conformally flat. These metrics are Kähler.

Reversing the sign on Λ^1 gives metrics on $\mathbb{S}^2 \times \mathbb{S}^2$.

2.2 Euclidean Schwarzschild

The Euclidean Schwarzshild metric was first written down by Gibbons-Hawking [5] in support of their Euclidean quantum gravity project. It is a straightforward Wick rotation of the classic 1915 Schwarzschild metric. The Euclidean Schwarzschild metric is

$$g \;=\; \frac{1}{1-\frac{2m}{r}}dr^2+\left(1-\frac{2m}{r}\right)d\psi^2+r^2\left(d\theta^2+\sin^2\theta\,d\varphi^2\right). \tag{4}$$

for $r\in(2m,\infty)$, $\psi\in[0,8m\pi)$. There is a coordinate singularity at $r=2m$; to see how this is resolved, we focus on the "fiber" part of the metric, and make the substitution $\rho^2=8m(r-2m)$ to get

$$\begin{aligned} g_{fiber} \;&=\; \left(\frac{1}{16m^2}\rho^2+1\right)d\rho^2+\frac{\rho^2}{\rho^2+16m^2}d\psi^2 \\ &=\; \left(1+O(\rho^2)\right)d\rho^2+\frac{1}{16m^2}\left(\rho\,+\,O(\rho^3)\right)^2d\psi^2 \end{aligned} \tag{5}$$

Therefore as ρ nears 0, the metric looks like $d\rho^2+\rho^2d\left(\psi/4m\right)^2$, which is smooth (not cone-like) provided the range of $\frac{\psi}{4m}$ is $[0,2\pi)$, or $\psi\in[0,8m\pi)$.

Topologically, this metric exists on $\mathbb{S}^2\times\mathbb{R}^2$, with a clear "zero section" at $r=2m$ where the metric has a bolt-like resolution: a 2-sphere of volume $2m$.

Euclidean Schwarzschild is never Kähler. (There is a different family of Kähler scalar-flat metrics on $\mathbb{S}^2\times\mathbb{R}^2$, discussed elsewhere.)

Curvature and asymptotics

This metric is Ricci-flat. Its Weyl curvatures are

$$W^+ \;=\; \frac{m}{2r^3}\left(3\omega\otimes\omega-2Id_{\bigwedge^+}\right), \quad W^- \;=\; \frac{m}{2r^3}\left(3\omega'\otimes\omega'-2Id_{\bigwedge^-}\right). \tag{6}$$

where $\omega=dr\wedge d\psi+r^2\sin\theta\,d\theta\wedge d\varphi$, $\omega'=dr\wedge dt-r^2\cos\theta\,d\theta\wedge d\varphi$

Asymptotically we have $g\approx dr^2+d\psi^2+r^2(d\theta^2+\sin^2\theta\,d\varphi^2)$ so r is roughly the distance function. Therefore $|\,\mathrm{Rm}\,|=O(r^{-3})$ and volumes increase like $O(r^3)$.

2.3 Burns Metric

The Burns metric [1] appears to have come from a conference talk of Burns' and was popularized by Lebrun [6] [7] [8]. It can be written

$$\begin{aligned} g \;&=\; dr^2 \,+\, \left(r^2+m^2\right)\left(\eta_X^2+\eta_Y^2\right) \,+\, r^2\,\eta_Z^2 \\ &=\; dr^2 \,+\, \frac{1}{4}\left(r^2+m^2\right)\left(d\theta^2+\sin^2\theta d\varphi^2\right) \,+\, \frac{1}{4}r^2\left(d\psi+\cos\theta\, d\varphi\right)^2 \end{aligned} \tag{7}$$

where $r \in [0,\infty)$ Of course there is a coordinate singularity at $r=0$. Unlike the Tab-bolt case, removing the coordinate singularity does not require any identifications. At $r=0$ we can restrict g to θ, φ, where its metrizes the 2-sphere of radius $m/2$. Then looking at the transverse direction, coordinatized by r, ψ, the metric is

$$g \;=\; dr^2+\frac{1}{4}r^2d\psi^2 \;=\;=\; dr^2+r^2\left(d\frac{\psi}{2}\right)^2 \tag{8}$$

which is a smooth metric on $\mathbb{R}^2$ provided $\psi/2 \in [0,2\pi)$, which indeed it is. Thus no identifications on the level-sets $r=r_0$ need be made, and the asymptotic topology of level sets is indeed $\mathbb{S}^3$.

Topologically, these considerations show that the Burns metric inhabits the tautological bundle over $\mathbb{C}P^1$, which is $\mathcal{O}(-1)$. The Burns metric is Kähler.

Curvature and asympotics

The Burns metric is scalar-flat and half-conformally flat, but not Ricci flat. The Ricci vector is

$$\mathcal{RV} \;=\; \frac{2m^2}{(r^2+m^2)^2}\left(-\eta^1,\eta^2,\eta^3,-\eta^4\right) \tag{9}$$

and the Weyl curvatures are

$$W^+ \;=\; 0, \quad W^- \;=\; \frac{m^2}{(r^2+m^2)^2}\left(3\omega'\otimes\omega' - 2Id_{\bigwedge^-}\right) \tag{10}$$

Note r is the distance to the "bolt" (the 2-sphere at $r=0$). Volumes of balls grow like $O(r^4)$ and curvature decays like $O(r^{-4})$. This metric is asymptotically Euclidean.

2.4 Eguchi-Hanson Metric

The Eguchi-Hanson metric was developed independently by Eguchi-Hanson [4] and Calabi [2] in 1979. The Eguchi-Hanson metric is

$$\begin{aligned} g \;&=\; \frac{1}{1-\frac{m^4}{r^4}}dr^2 \;+\; r^2\left(\eta_X^2+\eta_Y^2\right) \;+\; r^2\left(1-\frac{m^4}{r^4}\right)\eta_Z^2 \\ &=\; \frac{1}{1-\frac{m^4}{r^4}}dr^2 \;+\; \frac{1}{4}r^2\left(d\theta^2+\sin^2\theta\,d\varphi^2\right)^2 \;+\; \frac{1}{4}r^2\left(1-\frac{m^4}{r^4}\right)\left(d\psi+\cos\theta\,d\varphi\right)^2 \end{aligned} \tag{11}$$

for $r \in (m,\infty)$. To resolve the coordinate singularity at $r = m$, we examing the riber direction, parameterized by r, ψ, and with the substitution $r = \rho^2 + m$ we get

$$\begin{aligned} g \;&=\; m\left[\frac{\frac{4}{m}(\rho^2+m)^4}{(\rho^2+2m)(\rho^2+2m\rho+2m^2)}d\rho^2 + \frac{\frac{1}{4m}(\rho^2+2m)(\rho^2+2m\rho+2m^2)}{(\rho^2+m)^2}\rho^2 d\psi^2\right] \\ &=\; m\left[(1+O(\rho^2))d\rho^2 \;+\; (1+O(\rho^2))\rho^2 d\psi^2\right] \end{aligned} \tag{12}$$

This is smooth (that is, non-conical) provided $\psi \in [0, 2\pi)$. Because ψ ranges from 0 to 4π on $\mathbb{S}^3$, we must identify ψ with $\psi + 2\pi$ so that φ, θ, ψ now parameterizes $\mathbb{R}^3$.

Topologically, the Eguchi-Hanson metric exists on the cotangent bundle of $\mathbb{S}^2$, which is $O(-2)$.

These metrics are Kähler, with Kähler form

$$\omega = \frac{r}{\sqrt{1-\frac{m^4}{r^4}}}dr\wedge\eta_Y + r^2\sqrt{1-\frac{m^4}{r^4}}\eta_X\wedge\eta_Z. \tag{13}$$

Curvature and asymptotics

This metric is Ricci-flat and half-conformally flat. The Weyl tensors are

$$W^+ \;=\; 0, \quad W^- \;=\; \frac{2m^4}{r^6}\left(3\omega'\otimes\omega' - 2Id_{\bigwedge^-}\right). \tag{14}$$

Asymptotically we have $r \approx \mathit{dist}$. We get curvature decay $O(r^{-6})$ and Euclidean volume growth of $O(r^4)$. The Eguchi-Hanson metric is an ALE Kähler gravitational isntanton.

2.5 Lebrun metrics on $O(-k)$

Lebrun [8] generalized the Burns metric on $O(-1)$ and the Eguchi-Hanson metric on $O(-2)$ to scalar-flat Kähler ALE metrics on $O(-k)$ for all $k \in \{1, 2, 3, 4, \ldots\}$. The Lebrun metrics are

$$g \;=\; \frac{1}{\left(1-\frac{m^2}{r^2}\right)\left(1+\frac{(k-1)m^2}{r^2}\right)}dr^2 + r^2\left(\eta_X^2+\eta_Y^2\right) + r^2\left(1-\frac{m^2}{r^2}\right)\left(1+\frac{(k-1)m^2}{r^2}\right)\eta_Z^2 \quad (15)$$

If $k = 1$ this is the Burns metric (with the substitution $\rho = \sqrt{r^2 - m^2}$), and if $k = 2$ this is the Eguchi-Hanson metric. To resolve the coordinates singularity at $r = m$, we use $r = \rho^2 + m$ and examine the metric on the 2-plane parameterized by r, ψ:

$$\begin{aligned} g_{fiber} \;&=\; \frac{2m}{k}\left[\frac{\frac{2k}{m}(\rho^2+m)^4}{(\rho^2+2m)(\rho^4+2m\rho^2+km^2)}d\rho^2 + \frac{(\rho^2+2m)(\rho^4+2m\rho^2+km^2)}{2km(\rho^2+m)^2}\frac{k^2}{4}\rho^2 d\psi^2\right] \\ &=\; \frac{2m}{k}\left[(1+O(\rho^2))d\rho^2 \;+\; (1+O(\rho^2))\rho^2\left(d\,\frac{k}{2}\psi\right)^2\right] \end{aligned} \quad (16)$$

and so we require $\psi \in [0, \frac{4\pi}{k})$ in order for this metric to be smooth (not conical) near $\rho = 0$. Thus we make the identification $\psi \approx \psi + \frac{4\pi}{k}$, which makes the level-sets $r = const$ into lens spaces $L(k, 1)$.

Topologically, the Lebrun metric with parameter $k \in \{1, 2, \ldots\}$ is a metric on $O(-k)$. These metrics are always Kähler with Kähler form ω (see introduction).

Curvature and asymptotics

For each $k = 1, 2, \ldots$ these metrics are scalar-flat and half-conformally flat. They are Ricci-flat if and only if $k = 2$, which is the Eguchi-Hanson instanton. The Ricci vector is

$$\mathcal{RV} \;=\; \frac{2(k-2)m^2}{r^4}\left(\eta^1, -\eta^2, -\eta^3, \eta^4\right) \quad (17)$$

and the Weyl tensors are

$$W^+ \;=\; 0, \quad W^- \;=\; \frac{2(k-1)m^4 - (k-2)m^2r^2}{r^6}\left(3\omega'\otimes\omega' \;-\; Id_{\bigwedge^-}\right) \quad (18)$$

Curvature decay is $O(r^{-4})$ (except in the Eguchi-Hanson case $k = 2$ when decay is $O(r^{-6})$) and volume growth is $O(r^4)$. These metrics are ALE (asymptotically locally Euclidean) for all $k \in \mathbb{N}$; the metric in the case $k = 1$ is AE (asymptotically Euclidean).

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