Squashed 3-Spheres

Primer

Skip to a section:

1 Basics

The 3-sphere is special, not just among 3-manifolds but even among all sphere. This specialness comes these three facts, which we arrange from weakest to strongest:

- Round $\mathbb{S}^3$ is a parallelizable isotropic homogeneous space
- $\mathbb{S}^3$ has the structure of a simple Lie group
- $\mathbb{S}^3$ is the group of unit-length elements in a division algebra

We explore all three of these intertwined viewpoints on $\mathbb{S}^3$. As a natural subset of $\mathbb{R}^4$, we write down a framing, the stereographic coordinates, and the Euler coordinates (in §3.2).

The framing fields $\{X_1, X_2, X_3\}$ form the real Lie algebra $\mathfrak{su}(2)$, which is an expression of the fact $\mathbb{S}^3 \approx SU(2)$. This leads to the study of the Pauli matrices and the exponential map. We examine the covering map $Spin(3) \approx SU(2) \to SO(3)$. The theory of Riemannian metrics on Lie algebras leads to large classes of homogeneous metrics on $\mathbb{S}^3$. We observe the phenomenon of *collapse with bounded curvature*, seen first by Berger. We briefly examine the view of $\mathbb{S}^3$ as the unit sphere within the quaternions $\mathbb{H}$.

We have tried to include useful computational information, particaularly for students who may be trying to match up a sometimes outdated physics literature from decades ago with modern formulation. For instance transitions to and from frames and coordinates are given in §3.3 (in physics, and only in dimension 4, these are sometimes called the vierbein transitions).

The "squashed spheres" metrics, described generically in 4 and explicitly constructed later, are sometimes called cohomoegeity-1 Bianchi IX metrics. Bianchi-type metrics more generally are discussed elsewhere in these notes.

2 Topological $\mathbb{S}^3$ within $\mathbb{R}^4$

Skip to subsection:

The Framing — The fields on $\mathbb{R}^4$ and $\mathbb{S}^3$ and bracket relations

Connection and Curvature — Computation of θ and Ω

2.1 The Framing

Let x^1, x^2, x^3, x^4 be coordinates on $\mathbb{R}^4$, and let $r = \sqrt{(x^1)^2 + (x^2)^2 + (x^3)^2 + (x^4)^2}$. Consider the vector fields

$$\begin{aligned}
R &= x^1\frac{\partial}{\partial x^1} + x^2\frac{\partial}{\partial x^2} + x^3\frac{\partial}{\partial x^3} + x^4\frac{\partial}{\partial x^4} \\
X &= x^4\frac{\partial}{\partial x^1} + x^3\frac{\partial}{\partial x^2} - x^2\frac{\partial}{\partial x^3} - x^1\frac{\partial}{\partial x^4} \\
Y &= -x^3\frac{\partial}{\partial x^1} + x^4\frac{\partial}{\partial x^2} + x^1\frac{\partial}{\partial x^3} - x^2\frac{\partial}{\partial x^4} \\
Z &= x^2\frac{\partial}{\partial x^1} - x^1\frac{\partial}{\partial x^2} + x^4\frac{\partial}{\partial x^3} - x^3\frac{\partial}{\partial x^4}
\end{aligned} \tag{1}$$

with corresponding dual covector fields

$$\begin{aligned}
R_\flat &= x^1dx^1 + x^2dx^2 + x^3dx^3 + x^4dx^4 \;=\; R_\flat \\
X_\flat &= x^4dx^1 + x^3dx^2 - x^2dx^3 - x^1dx^4 \;=\; X_\flat \\
Y_\flat &= -x^3dx^1 + x^4dx^2 + x^1dx^3 - x^2dx^4 \;=\; Y_\flat \\
Z_\flat &= x^2dx^1 - x^1dx^2 + x^4dx^3 - x^3dx^4 \;=\; Z_\flat.
\end{aligned} \tag{2}$$

These are mutually orthogonal, have norms $|X| = |Y| = |Z| = |R| = r$. One notices that X, Y, Z are Killing fields and R is a gradient field: $R = \nabla\frac{1}{2}r^2$ and $\eta_0 = d(\frac{1}{2}r^2)$. We have the bracket relations

$$[X, Y] = 2Z, \quad [Z, X] = 2Y, \quad [Y, Z] = 2X, \quad [R, \cdot\,] \equiv 0. \tag{3}$$

Restricting to $\mathbb{S}^3 = \{r = 1\}$, the fields X, Y, Z are orthonormal and tangent to $\mathbb{S}^3$ and so constitute a framing of $\mathbb{S}^3$. The fields $\frac{1}{2}X$, $\frac{1}{2}Y$, $\frac{1}{2}Z$ under the Lie bracket constitute the algebra $\mathfrak{su}(2) = \mathfrak{so}(3)$:

$$\left[\frac{1}{2}X,\, \frac{1}{2}Y\right] \;=\; \frac{1}{2}Z \tag{4}$$

and cyclic permutations. Finally notice $d\eta^0 = 0$ and that restricted to $\mathbb{S}^3$

$$\begin{aligned}
dX_\flat &= -2(dx^1\wedge dx^4 \,+\, dx^2\wedge dx^3)\big|_{\mathbb{S}^3} \;=\; -2Y_\flat\wedge Z_\flat \\
dY_\flat &= 2\,(dx^1\wedge dx^3 \,-\, dx^2\wedge dx^4)\big|_{\mathbb{S}^3} \;=\; 2X_\flat\wedge Z_\flat \\
dZ_\flat &= -2\,(dx^1\wedge dx^2 \,+\, dx^3\wedge dx^4)\big|_{\mathbb{S}^3}, \;=\; -2X_\flat\wedge Y_\flat
\end{aligned} \tag{5}$$

In the context of $\mathbb{R}^4$ instead of $\mathbb{S}^3$, we see $span_{\mathbb{R}}\{dX_\flat, dY_\flat, dZ_\flat\} = \bigwedge{}^+$.

2.2 Connection and Curvature

Recall that we define connection 1-forms uniquely by θ by $d\eta^i = -\theta^i_j \wedge \eta^j$, $\theta^i_j = -\theta^j_i$; see the general primer. Using $\eta^1 = X_\flat$, $\eta^2 = Y_\flat$, $\eta^3 = Y_\flat$, from (5) we compute

$$\theta \;=\; \begin{pmatrix} 0 & -\eta^3 & \eta^2 \\ \eta^3 & 0 & -\eta^1 \\ -\eta^2 & \eta^1 & 0 \end{pmatrix} \tag{6}$$

From this we compute Riemannian curvature $\Omega = d\theta + \theta \wedge \theta$

$$\Omega \;=\; \begin{pmatrix} 0 & \eta^1 \wedge \eta^2 & \eta^1 \wedge \eta^3 \\ -\eta^1 \wedge \eta^2 & 0 & \eta^2 \wedge \eta^3 \\ -\eta^1 \wedge \eta^3 & -\eta^2 \wedge \eta^3 & 0 \end{pmatrix} \tag{7}$$

the Ricci curvature $\mathrm{Ric} = (\mathrm{Ric}_k)_{k=1}^3 = (i_{\eta^l}\Omega^l_k)_{k=1}^3$

$$\mathrm{Ric} \;=\; \left(2\eta^1,\, 2\eta^2,\, 2\eta^3\right) \tag{8}$$

and the scalar curvature $R = i_{\eta^i}\,\mathrm{Ric}_i$ is

$$R \;=\; 6. \tag{9}$$

From this, the Riemann tensor $\mathrm{Rm} = \mathrm{Rm}_{\,ijkl}{}^l$ is easily computable:

$$\mathrm{Rm}_{\,ijk}{}^l\, dx^i \otimes dx^j \;=\; \Omega^l_k. \tag{10}$$

We have for example $\mathrm{Rm}_{122}{}^1 = 1$ and $\mathrm{Rm}_{123}{}^1 = 0$.

This is a metric of constant sectional curvature $+1$.

3 $\mathbb{S}^3$ as a group

Skip to subsection:

3.1 Identification of $\mathbb{S}^3$ with $SU(2)$

The Lie group $SU(2)$ is the group of 2×2 matrices A such that $A\overline{A}^T = Id$ and $det(A) = 1$. Concretely,

$$SU(2) \;=\; \left\{ \; \left(\begin{array}{cc} z & -\bar{w} \\ w & \bar{z} \end{array} \right) \;\; \middle| \;\; |z|^2 + |w|^2 \;=\; 1 \; \right\}. \tag{11}$$

This is clearly the 3-sphere. Its Lie algebra is

$$\mathfrak{su}(2) = span_{\mathbb{R}}\{-\sqrt{-1}\sigma_X, -\sqrt{-1}\sigma_Y, -\sqrt{-1}\sigma_Z\} \tag{12}$$

where σ_X, σ_Y, σ_Z are the standard Pauli matrices

$$\sigma_X \;=\; \left(\begin{array}{cc} 0 & 1 \\ 1 & 0 \end{array} \right), \qquad \sigma_Y \;=\; \left(\begin{array}{cc} 0 & -\sqrt{-1} \\ \sqrt{-1} & 0 \end{array} \right), \qquad \sigma_Z \;=\; \left(\begin{array}{cc} 1 & 0 \\ 0 & -1 \end{array} \right) \tag{13}$$

which are the trace-free Hermitian matrics. We choose three distinguished $\mathbb{S}^1$ subgroups, each parameterized by $\mathbb{R}^1$:

$$\begin{gathered} \gamma_X(t) \;=\; \left(\begin{array}{cc} \cos(t) & -\sqrt{-1}\sin(t) \\ -\sqrt{-1}\sin(t) & \cos(t) \end{array} \right), \qquad \gamma_Y(t) \;=\; \left(\begin{array}{cc} \cos(t) & -\sin(t) \\ \sin(t) & \cos(t) \end{array} \right) \\ \gamma_Z(t) \;=\; \left(\begin{array}{cc} e^{-\sqrt{-1}t} & 0 \\ 0 & e^{\sqrt{-1}t} \end{array} \right). \end{gathered} \tag{14}$$

Clearly the derivatives are the basis vectors of $\mathfrak{su}(2)$:

$$X \triangleq \dot{\gamma}_X(0) = -\sqrt{-1}\sigma_X, \quad Y \triangleq \dot{\gamma}_Y(0) = -\sqrt{-1}\sigma_Y, \quad Z \triangleq \dot{\gamma}_Z(0) = -\sqrt{-1}\sigma_Z, \tag{15}$$

and we have the $\mathfrak{su}(2)$ relations $[X, Y] = 2Z$ and cyclic permutations. Compare (3).

Each of the distinguished subgroups acts on $SU(2)$ on the right; taking the derivatives of these actions gives a corresponding left-invariant field. For instance $X = -\sqrt{-1}\sigma_X$ gives

$$\left(\begin{array}{cc} x^1 + \sqrt{-1}x^2 & -x^3 + \sqrt{-1}x^4 \\ x^3 + \sqrt{-1}x^4 & x^1 - \sqrt{-1}x^2 \end{array} \right) \left(\begin{array}{cc} 0 & -\sqrt{-1} \\ -\sqrt{-1} & 0 \end{array} \right) \tag{16}$$

$$= \left(\begin{array}{cc} x^4 + \sqrt{-1}x^3 & x^2 - \sqrt{-1}x^1 \\ -x^2 - \sqrt{-1}x^1 & -x^4 - \sqrt{-1}x^3 \end{array} \right) \tag{17}$$

which is the vector field on $\mathbb{S}^3 \subset \mathbb{R}^4$ expressed by

$$x^4 \frac{\partial}{\partial x^1} + x^3 \frac{\partial}{\partial x^2} - x^2 \frac{\partial}{\partial x^3} - x^1 \frac{\partial}{\partial x^4}, \tag{18}$$

which is the vector field X from (3). Similarly the fields Y, Z from (3) are obtained by the action of the Lie algebra matrices $Y = -\sqrt{-1}\sigma_Y$ and $Z = -\sqrt{-1}\sigma_Z$.

3.2 Euler Coordinates

Consider the three distinguished subgroups from §2.2, given by

$$\gamma_X(t) = e^{-t\sqrt{-1}\sigma_X}, \quad \gamma_Y(t) = e^{-t\sqrt{-1}\sigma_Y}, \quad \gamma_Z(t) = e^{-t\sqrt{-1}\sigma_Z}. \tag{19}$$

Using just γ_Z and γ_Y we can parameterize $\mathbb{S}^3$ by

$$(\varphi,\theta,\psi) \mapsto \gamma_Z\left(\frac{\varphi}{2}\right)\gamma_Y\left(\frac{\theta}{2}\right)\gamma_Z\left(\frac{\psi}{2}\right) = \begin{pmatrix} \cos(\theta/2)\, e^{-\frac{\sqrt{-1}}{2}(\psi+\varphi)} & -\sin(\theta/2)\, e^{\frac{\sqrt{-1}}{2}(\psi-\varphi)} \\ \sin(\theta/2)\, e^{-\frac{\sqrt{-1}}{2}(\psi-\varphi)} & \cos(\theta/2)\, e^{\frac{\sqrt{-1}}{2}(\psi+\varphi)} \end{pmatrix} \tag{20}$$

The reason for the factor of $1/2$ has to do with the 2-to-1 spin cover $SU(2) \to SO(3)$. The ranges are $\theta \in [0, 2\pi)$, $\varphi, \psi \in [0, 4\pi)$. The triple (φ, θ, ψ) are called the Z-Y-Z Euler coordinates (the Wikipedia article on Euler coordinates is recommended), and are called the *rotation*, *nutation*, and *procession* variables, repsectively. From the embedding $\mathbb{S}^3 \hookrightarrow \mathbb{R}^4$ we obtain rectangular coordinates:

$$\begin{aligned}(\varphi,\theta,\psi) \;\longmapsto\; \Bigg(&\cos\left(\frac{\theta}{2}\right)\cos\left(\frac{\psi+\varphi}{2}\right), -\cos\left(\frac{\theta}{2}\right)\sin\left(\frac{\psi+\varphi}{2}\right), \\ &\sin\left(\frac{\theta}{2}\right)\cos\left(\frac{\psi-\varphi}{2}\right), -\sin\left(\frac{\theta}{2}\right)\sin\left(\frac{\psi-\varphi}{2}\right)\Bigg)\end{aligned} \tag{21}$$

and the resulting coordinate fields spanning $T_p\mathbb{S}^3 \subset T_p\mathbb{R}^4$ (most $p \in \mathbb{S}^3 \subset \mathbb{R}^4$) are

$$\begin{aligned}
\frac{\partial}{\partial\varphi} &= \frac{1}{2}\left(x^2\frac{\partial}{\partial x^1} - x^1\frac{\partial}{\partial x^2} - x^4\frac{\partial}{\partial x^3} + x^3\frac{\partial}{\partial x^4}\right) = \frac{1}{2}\overline{Z} \\
\frac{\partial}{\partial\theta} &= \sqrt{\frac{(x^3)^2+(x^4)^2}{(x^1)^2+(x^2)^2}}\left(-\frac{x^1}{2}\frac{\partial}{\partial x^1} - \frac{x^2}{2}\frac{\partial}{\partial x^2}\right) + \sqrt{\frac{(x^1)^2+(x^2)^2}{(x^3)^2+(x^4)^2}}\left(\frac{x^3}{2}\frac{\partial}{\partial x^3} + \frac{x^4}{2}\frac{\partial}{\partial x^4}\right) \\
\frac{\partial}{\partial\psi} &= \frac{1}{2}\left(x^2\frac{\partial}{\partial x^1} - x^1\frac{\partial}{\partial x^2} + x^4\frac{\partial}{\partial x^3} - x^3\frac{\partial}{\partial x^4}\right) = \frac{1}{2}Z
\end{aligned} \tag{22}$$

where $\bar{Z}$ is the right-invariant version of the left-invariant field Z; sadly the nutation field $\frac{\partial}{\partial\theta}$ belies simpler description. In these coordinates, the round metric is

$$g = \frac{1}{4}\begin{pmatrix} 1 & 0 & \cos\theta \\ 0 & 1 & 0 \\ \cos\theta & 0 & 1 \end{pmatrix}. \tag{23}$$

A second common set of coordinate is stereographic coordinates. Projecting stereographically from (x^1, x^2, x^3, x^4) to (x, y, z), we obtain a chart mapping $\mathbb{S}^3 \backslash \{pt\}$ onto $\mathbb{R}^3$. The pushforward metric is conformal to Euclidean: it is $4(1+(x)^2+(y)^2+(z)^2)^{-2}$ times the Euclidean metric of $\mathbb{R}^3$.

3.3 Vierbein transisitons

We have polar coordinate on $\mathbb{R}^4$ given by $(r, \varphi, \theta, \psi)$ were

$$\begin{aligned} x^1 &= r\cos(\theta/2)\cos\left(\frac{\varphi+\psi}{2}\right), & x^2 &= -r\cos(\theta/2)\sin\left(\frac{\varphi+\psi}{2}\right) \\ x^3 &= r\sin(\theta/2)\cos\left(\frac{\varphi-\psi}{2}\right), & x^4 &= -r\sin(\theta/2)\sin\left(\frac{\varphi-\psi}{2}\right). \end{aligned} \tag{24}$$

For convenience we use auxiliary functions $r^1 = \sqrt{(x^1)^2 + (x^2)^2}$, $r^2 = \sqrt{(x^3)^2 + (x^4)^2}$. We have the fields (22) and differentials

$$\begin{aligned} dr &= \frac{x^1}{r}dx^1 + \frac{x^2}{r}dx^2 + \frac{x^3}{r}dx^3 + \frac{x^4}{r}dx^4 \\ d\varphi &= \frac{x^2}{(r^1)^2}dx^1 - \frac{x^1}{(r^1)^2}dx^2 - \frac{x^4}{(r^2)^2}dx^3 + \frac{x^3}{(r^2)^2}dx^4, \\ d\theta &= -2\frac{r^2}{r^1}\left(x^1dx^1 + x^2dx^2\right) + 2\frac{r^1}{r^2}\left(x^3dx^3 + x^4dx^4\right), \\ d\psi &= \frac{x^2}{(r^1)^2}dx^1 - \frac{x^1}{(r^1)^2}dx^2 + \frac{x^4}{(r^2)^2}dx^3 - \frac{x^3}{(r^2)^2}dx^4. \end{aligned} \tag{25}$$

Using $\cos(\theta) = (r^1)^2 - (r^2)^2$, $1 = (r^1)^2 + (r^2)^2$, and $2r_1r_2 = \sin\theta$ one computes

$$\begin{aligned} 2X_\flat &= -\sin(\psi)\,d\theta \;+\; \sin(\theta)\cos(\psi)\,d\varphi, \\ 2Y_\flat &= \cos(\psi)\,d\theta \;+\; \sin(\theta)\sin(\psi)\,d\varphi, \\ 2Z_\flat &= d\psi + \cos(\theta)d\varphi, \\ X_\flat\otimes X_\flat \;+\; Y_\flat\otimes Y_\flat &= \frac{1}{4}\left((d\theta)^2 \;+\; \sin^2(\theta)\,(d\varphi)^2\right). \end{aligned} \tag{26}$$

What's happening is that the metric $(d\theta^2 + \sin^2\theta(d\varphi)^2)/4$, which is clearly a metric on a round 2-sphere of half radius, is the pullback 2-form on $\mathbb{S}^3$ of the standard half-radius metric under the Hopf fibration map $\mathbb{S}^3 \to \mathbb{S}^2$. The distribution along the fiber direction is spanned by $Z_\flat$, which is not itself a coordinate field. The round metric of curvature $+1$, in spherical coordinates, is

$$g \;=\; \frac{1}{4}\left(d\psi + \cos(\theta)d\varphi\right)^2 \;+\; \frac{1}{4}\left((d\theta)^2 + \sin^2(\theta)(d\varphi)^2\right) \tag{27}$$

In some applications, one sees the expression $d\psi + n\cos(\theta)d\varphi$. This cofield (and its dual vector field) have infinite singularities on the two principal Hopf circles unless $n = 1$.

3.4 Squashed spheres

Following the discussion in the primer on homogeneous spaces, once we have any Lie group G with algebra $\mathfrak{g} = T_eG$ and basis $\{v_1, \ldots, v_n\}$ for $\mathfrak{g}$, we can parallelize G by left-translating the fields $v_1, \ldots, v_n$ to all points of G. We then give G a metric by declaring these fields to be orthonormal. If $\{\eta^1, \ldots, \eta^n\}$ is the coframe to $\{v^1, \ldots, v^n\}$ then this metric on G is simply

$$g \;=\; \eta^1 \otimes \eta^1 \;+\; \ldots \;+\; \eta^n \otimes \eta^n. \tag{28}$$

Thus we can change the metric on G non-conformally by just changing the basis and declaring it to be orthonormal. Of course $n = 3$ in our case.

In the case of the sphere, let $\delta > 0$ and declare the frame $\{\delta^{-1}X, Y, Z\}$ to be orthonormal; this gives a new metric g_δ. If $\{\eta^1, \eta^2, \eta^3\}$ is the coframe, one computes

$$d\eta^1 \;=\; -2\delta\eta^2 \wedge \eta^3, \quad d\eta^2 \;=\; 2\delta^{-1}\eta^1 \wedge \eta^3, \quad d\eta^3 \;=\; -2\delta^{-1}\eta^1 \wedge \eta^2 \tag{29}$$

which gives the connection 1-form

$$\theta \;=\; \begin{pmatrix} 0 & -\delta\eta^3 & \delta\eta^2 \\ \delta\eta^3 & 0 & (\delta - 2\delta^{-1})\eta^1 \\ -\delta\eta^2 & -(\delta - 2\delta^{-1})\eta^1 & 0 \end{pmatrix} \tag{30}$$

the curvature 2-form

$$\Omega \;=\; \begin{pmatrix} 0 & \delta^2\eta^1 \wedge \eta^2 & \delta^2\eta^1 \wedge \eta^3 \\ -\delta^2\eta^1 \wedge \eta^2 & 0 & (-3\delta^2 + 4)\eta^2 \wedge \eta^3 \\ -\delta^2\eta^1 \wedge \eta^3 & (3\delta^2 - 4)\eta^2 \wedge \eta^3 & 0 \end{pmatrix} \tag{31}$$

the Ricci vector

$$\mathcal{RC} \;=\; \big(2\delta^2\eta^1,\, (4 - 2\delta^2)\eta^2,\, (4 - 2\delta^2)\eta^3\big) \tag{32}$$

and the scalar curvature $R = 8 - 2\delta^2$. As $\delta \to 0$ the metric singularizes but all curvature tensors remain bounded; this is the phenomenon of *collapse with bounded curvature*, first discovered by Berger. In the standard framing, these metrics are

$$g_\delta \;=\; \delta^2 X_\flat \otimes X_\flat + Y_\flat \otimes Y_\flat + Z_\flat \otimes Z_\flat \tag{33}$$

and are called the Berger metrics. Geometrically, as $\delta \to 0$ the vector field X is being scaled everwhere to become very small, while the sizes of Y and Z are left alone. This collapses the injectivity radius at all points to 0. Intuitively, as δ goes from 1 to 0, we are crushing a 3-sphere of radius 1 to a 2-sphere of radius 1/2. The remarkable aspect of collapse procedure is the boundedness of sectional curvatures.

3.5 The Quaternions

The group action on $\mathbb{S}^3$ is quaternionic multiplication.

Definition. The *quaternions* are the constituents of the algebra $\mathbb{H} = span_{\mathbb{R}}\{1, i, j, k\}$, where 1 is the identity and the algebra relations are $i^2 = j^2 = k^2 = ijk = -1$.

The algebra bracket $[a, b] = ab - ba$ makes the purely imaginary quaterions $span_{\mathbb{R}}\{i, j, k\}$ into a simple Lie algebra; we have the relations $[i, j] = 2k$ and cyclic permutations, which is the same as (3).

The quaterions have an inner product: $\langle q_1, q_2\rangle = q_1\overline{q_2}$ where the overline is quaternionic conjugation. One verifies that this is the usual Euclidean product under the natural identitifation $\mathbb{H} \approx \mathbb{R}^4$. The 3-sphere can therefore be considered the set of unit-norm quaternions. Similarly there is a natural inner producton $\mathbb{H}^n$; the algebra of automorphism that leave the $\mathbb{H}^n$ inner product intact is called $Sp(n)$.

Becuase $\langle q_1p, q_2\rangle = \langle q_1, \bar{p}q_2\rangle$ for any quaternion p, we easily verify that multiplicaiton by unit quaternions preserves the sphere. Thus a product $\mathbb{S}^3 \oplus \mathbb{S}^3 \to \mathbb{S}^3$ exists, given simply by quaternionic multiplication. Certainly also $\mathbb{S}^3$ acts on $\mathbb{H}$ via quaternionic multiplication; it is easily checked that $\mathbb{S}^3 \approx Sp(1)$.

Viewing $\mathbb{S}^3$ as a Lie group with quaternionic multiplication, one can check that it Lie algebra is the purely imaginary quaternions with the algebra bracket.

Definition. (Cayley-Dickson) If $\mathcal{A}$ is an algebra with involution $* : \mathcal{A} \to \mathcal{A}$, $*^2 = Id$, then $\mathcal{A} \oplus \mathcal{A}$ is an algebra with involution $(p, q)^* = (p^*, -q)$ and product

$$(p, q) \cdot (r, s) \ = \ (pr + s^*q,\ sp + qr^*) \tag{34}$$

We have algebra isomorphisms $\mathbb{C} = \mathbb{R} \oplus \mathbb{R}$ with the Cayley-Dickson product and $\mathbb{H} = \mathbb{C} \oplus \mathbb{C}$ with the Cayley-Dickson product. Goung further, the Octonions $\mathbb{O}$ are $\mathbb{H} \oplus \mathbb{H}$ with the Cayley-Dickson product; although $\mathbb{O}$ is non-associative, it is alternate-associative and retains its status as a division algebra—in particular there are no zero-divisors. The next algebra is the 16-dimensional Sedenions, which unfortunately are quite generic: the are not associative or even alternate-associative (although they are *flexible*: $a(ba) = (ab)a$), and have zero divisors so cannot be a division algebra. One may continue the Cayley-Dickson construction indefinitely, obtaining non-associative, non-division algebras in every dimension $n = 2^k$.

The quaternionic product on $\mathbb{S}^2$ is isomorphic with the $SU(2)$ product; this is an expression of the "low dimension accident" that $Sp(1) \approx SU(2)$.

4 Metrics on 4-manifolds

Many metrics on 4-manifolds are be formed on a cross product $\mathbb{R}^+\times\mathbb{S}^3$ or $\mathbb{R}\times\mathbb{S}^3$ using the generalized warped product construction we describe here. If r parametrizes the $\mathbb{R}^+$ or $\mathbb{R}$ factor, then 1-variable functions $f, g, h > 0$ determine a metric

$$g \;=\; A(r)^2 dr\otimes dr \;+\; B(r)^2 X_\flat\otimes X_\flat \;+\; C(r)^2 Y_\flat\otimes Y_\flat \;+\; D(r)^2 Z_\flat\otimes Z_\flat. \tag{35}$$

This is an example of a larger family of cohomogeneity-1 metrics, where smoothly varying left-invariant metrics on a Lie group are parameterized by a radial variable; in this case the Lie group is $\mathbb{S}^3 = SU(2)$. Referencing the Bianchi classification of Lie groups [1], this is sometimes called a metric with Bianchi type IX symmetry.

Simple examples with $A(r) = 1$ include flat Euclidean space $B(r) = C(r) = D(r) = r$, the 4-sphere $B(r) = C(r) = D(r) = \sin(r)$, and hyperbolic space $B(r) = C(r) = D(r) = \sinh(r)$ (or e^r or $\cosh(r)$).

Computations are easiest on frames. It is typical to label frames from 0 to 3 instead of 1 to 4; this is a legacy of the involvement of physics in the early stages both of studying applications of $SU(2)$ and spin groups, as well as early metric constructions such as the Lorentzian Taub-NUTs.

We have a frame $\sigma^0 = \frac{1}{A(r)}\nabla r$, $\sigma^1 = \frac{1}{B(r)}X$, $\sigma^2 = \frac{1}{C(r)}Y$, $\sigma^3 = \frac{1}{D(r)}Z$ and coframe $\eta^0 = A(r)dr$, $\eta^1 = B(r)X_\flat$, $\eta^2 = C(r)X_\flat$, $\eta^3 = D(r)Z_\flat$.

Working with $d\eta^i$, we compute the connection:

$$\theta \;=\; \begin{pmatrix} 0 & -\frac{1}{A}(\log B)'\eta^1 & -\frac{1}{A}(\log C)'\eta^2 & -\frac{1}{A}(\log D)'\eta^3 \\ \frac{1}{A}(\log B)'\eta^1 & 0 & \frac{-B^2-C^2+D^2}{BCD}\eta^3 & \frac{B^2-C^2+D^2}{BCD}\eta^2 \\ \frac{1}{A}(\log C)'\eta^2 & \frac{B^2+C^2-D^2}{BCD}\eta^3 & 0 & \frac{B^2-C^2-D^2}{BCD}\eta^1 \\ \frac{1}{A}(\log D)'\eta^3 & \frac{-B^2+C^2-D^2}{BCD}\eta^2 & \frac{-B^2+C^2+D^2}{BCD}\eta^1 & 0 \end{pmatrix}. \tag{36}$$

The curvature 2-form is too big to be usefully written down generically.

5 Exercises

1. Verify the computational claims made in this primer.

2. An aspect of $\mathbb{S}^3$ we did not discuss is its nature as a spin group A 2-to-1 group homomorphism $SU(2) \to SO(3)$, which is also a topological covering map, makes $SU(2)$ is the *spin group* associated to $SO(3)$, known as $Spin(3)$. It is well known that $SO(3)$ is topologically $\mathbb{R}P^3$.

   Obtain this 2-to-1 homomorphism as follows:

   (a) We certianly have $\mathbb{R}^3 \approx span_{\mathbb{R}}\{\sigma_X, \sigma_Y, \sigma_Z\}$. Show that $-det : \mathbb{R}^3 \to \mathbb{R}$ is a Euclidean metric, as $-det(\alpha\sigma_X + \beta\sigma_Y + \gamma\sigma_Z) = \alpha^2 + \beta^2 + \gamma^2$. Write down the polarization of $-det$ (that is, $-det$ as a *symmetric* bilinear form).

   (b) If $U \in SU(2)$ and $x \in \mathbb{R}^3$, verify that the action $U.x = Ux\overline{U}^T$ gives a group homomorphism $SU(2) \mapsto SO(3)$.

   (c) Show that the kernel of this group homomorphism is $\{Id, -Id\} \subset SU(2)$ and that the map is onto. Conclude that $SU(2) \mapsto SO(3)$ is a topological double cover of $SO(3)$.

   (d) The standard *principal rotations* in $SO(3)$, about the X, Y, and Z axes, repsectively, are

$$
\begin{aligned}
T_X(\alpha) &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha & -\sin\alpha \\ 0 & \sin\alpha & \cos\alpha \end{pmatrix}, \\
T_Y(\alpha) &= \begin{pmatrix} \cos\alpha & 0 & \sin\alpha \\ 0 & 1 & 0 \\ -\sin\alpha & 0 & \cos\alpha \end{pmatrix}, \\
T_X(\alpha) &= \begin{pmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & cos\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix}.
\end{aligned}
\tag{37}
$$

   Under the homomorphism just described, show that the path $\gamma_x(\alpha)$ in $SU(2)$ maps to the double-speed path $T_X(2\alpha)$ in $SO(3)$. Show similarly that $\gamma_Y(\alpha)$ maps to $T_Y(2\alpha)$ and $\gamma_Z(\alpha)$ maps to $T_Z(2\alpha)$.

References

[1] L. Bianchi “Lezioni sulla teoria dei gruppi continui niti di transformazioni,” Enrico Spoerri, 1918.