The Taub-NUT family of metrics

Skip to a section:

1 Introduction

A *Euclidean Taub-NUT style metric* is a metric of the form

$$g \;=\; \frac{1}{U(r)}(dr)^2 \;+\; V(r)\left((X_\flat)^2+(Y_\flat)^2\right) \;+\; m^2U(r)(Z_\flat)^2 \tag{1}$$

where $U(r)$ are smooth 1-variable functions and μ is a constant. Each slice $r = const$ is a 3-sphere with a left-invariant metrics. For cofields X, Y, Z, see the primer. The standard Euclidean Taub-NUT is the metric on $\mathbb{R}^4$ given by

$$g \;=\; \frac{1}{4}\frac{r+m}{r-m}(dr)^2+(r^2-m^2)\left((X_\flat)^2+(Y_\flat)^2\right)+4m^2\frac{r-m}{r+m}(Z_\flat)^2 \tag{2}$$

for $r > m$. There is an apparent singularity at $r = m$; this is the so-called "nut," a term for an isolated fixed point of each of the three Killing fields.

This metric comes from physics. The Lorentzian versions of these metrics occur via *Wick rotation*: substituting $\sqrt{-1}r$ for r (and one must also make an artifical substitution $\sqrt{-1}rm \mapsto rm$). In this Lorentzian version, the "nut" at $r = m$ becomes an event horizon, and the metric is smooth for $r \in (-\infty, \infty)$—the coordinate singularity can be removed using a version of the Kruskal trick [6]. The topology is $\mathbb{R} \times \mathbb{S}^3$, not $\mathbb{R}^4$.

The original Taub-NUT metric, which was Lorenzian, was partly discovered in Taub's 1951 exploration of cohomogeneity 1 space-time metrics [12] with space-like symmetries. On crossing the event horizon at $r = m$ one of the symmetries becomes time-like, so Taub missed this part of the metric. The Newman-Tamburino-Unti 1963 paper [9] studied the full metric, and some highly troubling phenomena were noticed such as the existence of closed time-like curves and the ability to complete the space across the event horizon in more than one way—see for example [7]. We will not further explore the Lorentzian versions of these metrics.

The Euclidean Taub-NUT metric was first written down by Hawking [5] in support of his Euclidean quantum gravity project. At present many interesting metrics fall under the heading of generalized Taub-NUT metrics, and many of these metrics can be arrived at in numerous ways, for instance by using Gibbon-Hawking ansatz or the toric Kähler ansatz, which is a specialization of the Joyce ansatz.

Definitions. A complete Ricci-flat, half conformally flat 4-manifold is often called a *gravitational instanton*. An open manifolds with curvature falloff $o(r^{-2})$ and asymptotic volume growth $O(r^4)$ it is called ALE; if it has curvature falloff $O(r^{-2})$ and asymptotic volume growth $O(r^3)$ it is called ALF.

2 Conventions

2.1 Conventions on frames and the Ricci vector

Our metrics here are written down on susbets of $\mathbb{R}\times\mathbb{S}^3$ or on a quotient of a subset of $\mathbb{R}\times\mathbb{S}^3$. Let us recall the primary objects on $\mathbb{S}^3$. We have three principal cofields

$$\begin{aligned} \eta_X &= x^4dx^1 + x^3dx^2 - x^2dx^3 - x^1dx^4 \\ \eta_Y &= -x^3dx^1 + x^4dx^2 + x^1dx^3 - x^2dx^4 \\ \eta_Z &= x^2dx^1 - x^1dx^2 - x^4dx^3 - x^3dx^4. \end{aligned} \tag{3}$$

If we restrict these to the unit sphere $\mathbb{S}^3 \subset \mathbb{R}^4$ we compute $d\eta_X = -2\eta_Y\wedge\eta_X$ and cyclic permutations, and we have $|\eta_X|^2 = |\eta_Y|^2 = |\eta_Z|^2 = 1$.

Then recall the Euler coordinates (φ,θ,ψ) on $\mathbb{R}^3$, where $\theta\in[0,2\pi)$ and $\varphi,\psi\in[0,4\pi)$. The range of ψ in particular plays a delicate role in the Taub-bolt metric. In fact ψ is more closely tied to the coframe $\{\eta_X,\eta_Y,\eta_Z\}$ than the other two variables, in the sense that in the coordinate frame $\{\frac{\partial}{\partial\varphi},\frac{\partial}{\partial\theta},\frac{\partial}{\partial\psi}\}$ we have

$$\frac{\partial}{\partial\psi} = (\eta_X)^\sharp; \tag{4}$$

recall this from the "squashed spheres primer" where we wrote $\frac{\partial}{\partial\psi} = Z$. We use η_X^2 to mean $\eta_X\otimes\eta_X$, etc. We have the relations

$$\eta_X^2 + \eta_Y^2 = \frac{1}{4}\left(d\theta^2 + \sin^2\theta\, d\varphi^2\right) \quad \text{and} \quad \eta_Z = d\psi + \cos\theta\, d\varphi. \tag{5}$$

The metrics here are $g = A^2dr^2 + B^2\eta_X^2 + B^2\eta_Y^2 + C^2\eta_Z^2$ and A, B, C, D are functions of one fvariable, and we have unit coframe

$$\eta^0 = A(r)dr, \quad \eta^1 = B(r)\eta_X, \quad \eta^2 = C(r)\eta_Y, \quad \eta^3 = D(r)\eta_Z. \tag{6}$$

In a few places we mention the "Ricci vector," so let us recall what this is. This comes from the Cartan formalism, where curvature is expressed as a matrix of 2-forms: $\Omega = (\Omega^i_j)$. It is natural to regard the Ricci tensor as a vector of 1-forms:

$$\mathcal{RV} = \left(i_{\eta_i}\Omega^i_j\right)_{j=1}^4. \tag{7}$$

For example, the Ricci vector of an Einstein metric has the form

$$\mathcal{RV} = \left(\Lambda\eta^0,\, \Lambda\eta^1,\, \Lambda\eta^2,\, \Lambda\eta^3\right). \tag{8}$$

2.2 Sections of $\bigwedge^{\pm}$

We establish conventions on section of $\bigwedge^+$ and $\bigwedge^-$, as well as $\bigwedge^+ \oplus \bigwedge^+$ and $\bigwedge^- \oplus \bigwedge^-$. We normally use unit coframes η^0, η^1, η^2, η^3, with the orientation that puts these in order. We have

$$\begin{aligned} \omega &= \eta^0 \wedge \eta^3 + \eta^1 \wedge \eta^2 \\ \eta &= \eta^0 \wedge \eta^2 - \eta^1 \wedge \eta^3 \\ \mu &= \eta^0 \wedge \eta^1 + \eta^2 \wedge \eta^3 \end{aligned} \tag{9}$$

as an orthogonal basis of $\bigwedge^+$. The exterior norms of these are all $\sqrt{2}$, and the tensor norms are all 2 (see the general primer for the difference between these). We use

$$\begin{aligned} \omega' &= \eta^0 \wedge \eta^3 - \eta^1 \wedge \eta^2 \\ \eta' &= \eta^0 \wedge \eta^2 + \eta^1 \wedge \eta^3 \\ \mu' &= \eta^0 \wedge \eta^1 - \eta^2 \wedge \eta^3 \end{aligned} \tag{10}$$

as our orthogonal basis of $\bigwedge^-$. (Apologies for re-using the symbol "η." We are using the notation of [1].)

Some of our expressions below, particularly for Weyl tensors, will require a basis of $\bigwedge^+ \odot \bigwedge^+$ and $\bigwedge^- \odot \bigwedge^-$. These are both six dimensional spaces. In particular

$$\begin{aligned} Id_{\bigwedge^+} &= \frac{1}{2}\left(\omega \otimes \omega + \eta \otimes \eta + \mu \otimes \mu\right), \\ Id_{\bigwedge^-} &= \frac{1}{2}\left(\omega' \otimes \omega' + \eta' \otimes \eta' + \mu' \otimes \mu'\right). \end{aligned} \tag{11}$$

Recall the convention that $Id.\omega = \frac{1}{2} Id_{ij}{}^{kl}\omega_{kl}$, etc. The Derdzinski theorem [1] says that, in the case of a Kähler 4-manifold with orientation compatible with the complex structure, we have

$$W^+ = \frac{R}{24}\left(3\omega \otimes \omega - 2Id_{\bigwedge^+}\right) = \frac{R}{24}\left(2\omega \otimes \omega - \eta \otimes \eta - \mu \otimes \mu\right) \tag{12}$$

where R is scalar curvature.

Sometimes we must switch the orientation, which amounts to declaring $\eta^1, \eta^2, \eta^3, \eta^0$ to be the correct order. Then $\bigwedge^+ = span_{\mathbb{R}}\{\omega', \eta', \mu'\}$ and $\bigwedge^- = span_{\mathbb{R}}\{\omega, \eta, \mu\}$

3 The Taub-NUT family of metrics

3.1 The standard Taub-NUT

One of the most studied of all metrics, the classic Euclidean Taub-NUT metric can be expressed in frames or coordinates:

$$\begin{aligned} g \;&=\; \frac{1}{4}\frac{r+m}{r-m}(dr)^2 + (r^2-m^2)\left(\eta_X^2+\eta_Y^2\right) + 4m^2\frac{r-m}{r+m}\eta_Z^2 \\ &=\; \frac{1}{4}\frac{r+m}{r-m}(dr)^2 + \frac{1}{4}(r^2-m^2)\left(d\theta^2+\sin^2\theta\, d\varphi^2\right) + m^2\frac{r-m}{r+m}(d\psi\,+\,cos\theta\, d\varphi)^2, \end{aligned} \tag{13}$$

$r\in[0,\infty)$. The paramater m is just a scale parameter: substituting $r=r'm$ gives

$$g \;=\; m^2\left(\frac{1}{4}\frac{r'+1}{r'-1}(dr')^2 + ((r')^2-1^2)\left(\eta_X^2+\eta_Y^2\right) + 4\frac{r'-1}{r'+1}\eta_Z^2\right) \tag{14}$$

An apparent singularity occurs at $r=m$. But one can change to the unit radial $\rho(r)=\int_0^r \frac{1}{2}\sqrt{\frac{r+m}{r-m}}dr$, which is integrable and smooth at $r=0$ and $\rho=\sqrt{m}\sqrt{r-m}+O(r-m)$. Then one checks that as $\rho\searrow 0$ the coefficients on the frames go to zero like $\rho+O(\rho^2)$; thus the metric is smooth at $r=m$.

As $r\to\infty$ we see that

$$g \;\approx\; \frac{1}{4}(dr)^2 \;+\; r^2\left(\eta_X^2+\eta_Y^2\right) + 4m^2\eta_Z^2 \tag{15}$$

so that the hopf fiber approaches length $2m$ while the base $\mathbb{S}^2$ expands like r, so asymptotic spheres are very "squashed." Volumes of large balls expand like $O(\rho^3)$.

One checks that if the Kähler form is taken to be ω, the associated complex structure is integrable and indeed g is a Kähler metric.

<u>*Curvature and asymptotics*</u>

Taub-NUT space is Ricci-flat and half conformally flat. We have $scal=0$, $\mathring{\mathrm{Ric}}=0$. One finds that

$$W^+ \;=\; 0, \qquad W^- \;=\; \frac{4m}{(r+m)^3}\left(3\omega'\otimes\omega' - 2Id_{\bigwedge^-}\right). \tag{16}$$

Because the unit radial ρ is r to first order at large distances, we see $|\,\mathrm{Rm}\,|=|W^-|$ decays like $O(\rho^{-3})$ as $\rho\to\infty$. This metric is a Kähler ALF gravitational instanton.

3.2 Extended Taub-NUTs

First written down by Page [10], the extended Taub-NUT metrics—also called the Taub-NUT metrics with magnetic anomaly—are

$$g \;=\; \frac{1}{U(r)}(dr)^2 + (r^2-n^2)\left(\eta_X^2+\eta_Y^2\right) + n^2U(r)\eta_Z^2 \tag{17}$$

where

$$U(r) \;=\; 4\frac{r^2-2mr\,+\,n^2}{r^2-n^2} \;=\; 4\frac{(r-r_+)(r-r_-)}{(r-n)(r+n)} \tag{18}$$

where $r_\pm = m \pm \sqrt{m^2-n^2}$ and we require $0 \le n \le m$, $r \in [r_+, \infty)$. Choosing $n = m$ gives the standard Taub-NUT metric. To ensure no curvature singularities, one must also ensure $r_+ > n$.

Page calls m the "electric mass" and n the "magnetic mass;" but in light of modern understandings of Weyl curvature, perhaps we should call $m+n$ the electric mass and $m-n$ the magnetic mass.

These metrics are not Kähler unless $m = n$, which is the standard Taub-NUT.

Curvature and asymptotics

These spaces are Ricci-flat: $R = 0$ and $\mathring{\mathrm{Ric}} = 0$. The Weyl tensors are

$$\begin{aligned} W^+ \;&=\; \frac{2(m-n)}{(r-n)^3}\left(3\omega\otimes\omega - 2Id_{\bigwedge^+}\right) \\ W^- \;&=\; \frac{2(m+n)}{(r+n)^3}\left(3\omega'\otimes\omega' - 2Id_{\bigwedge^-}\right) \end{aligned} \tag{19}$$

If ρ is the unit radial, again we see $O(\rho^{-3})$ curvature falloff and $O(\rho^3)$ volume growth. We see curvature singularities at $r = n$, which is the reason for stipulating $r_+ > n$. These metrics are ALF.

3.3 Taub-NUT anti-deSitter

These metrics are Einstein with Einstein constant Λ for $\Lambda \leq 0$. They take the form

$$g \;=\; \frac{1}{U(r)}(dr)^2 + (r^2 - n^2)\left(\eta_X^2 + \eta_Y^2\right) + n^2 U(r)\eta_Z^2 \tag{20}$$

where

$$U(r) \;=\; \frac{4(3 + \Lambda n^2)(r^2 - 2mr\, +\, n^2) - \Lambda(r^2 - n^2)^2}{3(r^2 - n^2)}. \tag{21}$$

(This is the same U as in (23).) These are not Kähler unless $\Lambda = 0$, $m = n$, which is the standard Taub-NUT.

For these metrics to be non-singular and open, at least we must have $\Lambda < 0$, $3 + \Lambda n^2 > 0$, and $0 \leq n \leq m$. But the general question of when these metrics are non-singular is delicate. Here are the criteria: first the largest zero of U must be larger than n (this involves solving a 4-th order polynomial), and second at this zero we must have the quantity *diameter/radius* be $2\pi/s$ for some $s \in \{1, 2, \dots\}$.

Curvature and asymptotics

These metrics are Einstein with Einstein constant $\Lambda < 0$ and scalar curvature 4Λ. We have $\mathring{\mathrm{Ric}} \equiv 0$. The Weyl tensors are

$$\begin{aligned} W^+ \;&=\; \frac{2(m-n)(3+\Lambda n^2)}{3(r-n)^3}\left(3\omega \otimes \omega - 2Id_{\bigwedge^+}\right) \\ W^- \;&=\; \frac{2(m+n)(3+\Lambda n^2)}{(r+n)^3}\left(3\omega' \otimes \omega' - 2Id_{\bigwedge^-}\right). \end{aligned} \tag{22}$$

Clearly these are non-singular for $r > n$. If ρ is the unit radial, then $\rho = O(\log(r))$ as r distance gets large. We see that $|W^+|$, $|W^-|$ decay like $O((\log(\rho))^{-3/2})$. Volumes of metric balls expand exponentially: like $O(e^{3\rho})$. The full Riemann tensor does not decay, since Ric is a multiple of g.

3.4 Taub-NUT deSitter

These are the positive scalar curvature analogs of the Λ-Taub-NUT metrics. These metrics are Einstein with Einstein constant Λ for $\Lambda \geq 0$. They take the form

$$
\begin{aligned}
g &= \frac{1}{U(r)}(dr)^2 + (r^2 - n^2)\left(\eta_X^2 + \eta_Y^2\right) + n^2 U(r)\eta_Z^2, \\
U(r) &= \frac{4(r^2 - 2mr + n^2) + \Lambda(n^4 + 2n^2r^2 - \frac{1}{3}r^4)}{(r^2 - n^2)}.
\end{aligned}
\tag{23}
$$

(This is actually the same U as in (21).) These are always compact when $\Lambda > 0$, but these are not necessarily manifolds: the question of when coordinate singularities can be removed is delicate. There are two bolts, which result is two quantization conditions. The quartic numerator of U must have two positive roots r_+, r_{++}, both larger than n. At both roots, a bolt is attached. For the manifold to be smooth, there will be a quantization condition on ψ and another on the constant Λ.

Both quantization conditions can almost never be simultaneously met constently, so the metric 23 with $\Lambda > 0$ is almost never smooth. However Page [11] showed that both conditions can be simultaneously solved in one case, and produced the so-caled Page metric $\mathbb{C}P^2 \# \overline{\mathbb{C}P}^2$

These are never Kähler when $\Lambda > 0$, though the Page metric is conformally Kähler.

Curvature

These metrics are Einstein with Einstein constant Λ. The Weyl tensors are

$$
\begin{aligned}
W^+ &= \frac{2(3m - 3n - n^3\Lambda)}{3(r-n)^3}\left(3\omega \otimes \omega - 2Id_{\bigwedge^+}\right) \\
W^- &= \frac{2(3m + 3n + n^3\Lambda)}{3(r+n)^3}\left(3\omega' \otimes \omega' - 2Id_{\bigwedge^+}\right).
\end{aligned}
\tag{24}
$$

3.5 Taub-Bolt

The Taub-bolt metric is due to Page [10]. We set $m = 5n$ in extended Taub-NUT:

$$\begin{aligned} g &= \frac{1}{U(r)}(dr)^2 + (r^2-n^2)\left(\eta_X^2+\eta_Y^2\right) + n^2U(r)\,\eta_Z^2 \\ &= \frac{1}{U(r)}(dr)^2 + \frac{1}{4}(r^2-n^2)\left(d\theta^2+\sin^2\theta\,d\varphi^2\right) + \frac{1}{4}n^2U(r)(d\psi+\cos\theta\,d\varphi)^2 \end{aligned} \tag{25}$$

where $U(r) = 4\frac{r^2-\frac{5}{2}nr+n^2}{r^2-n^2} = 4\frac{(r-2n)(r-n/2)}{(r-n)(r+n)}$ and $r \in (2n,\infty)$.

The intersting part is what happens at $r = 2n$. The 2-sphere parameterized by θ, φ does not squeeze to a point, but converges to a round sphere of radius $\sqrt{3}n/2$. For each fixed θ,φ (so fixing a point on the base 2-sphere) the variables $r \in [2n,\infty)$, $\psi \in [0,4\pi)$ parameterize a copy of $\mathbb{R}^2$.

Restricting to the $\mathbb{R}^2$ part and making the change of variable $\rho = 2\sqrt{r-2n}$

$$\begin{aligned} g &= \frac{1}{4}\frac{\left(\frac{1}{4}\rho^2+n\right)\left(\frac{1}{4}\rho^2+3n\right)}{\frac{1}{4}\rho^2+\frac{3n}{2}}\left(dv^2 \;+\; \left(\frac{n\left(\frac{1}{4}\rho^2+\frac{3n}{2}\right)}{\left(\frac{1}{4}\rho^2+n\right)\left(\frac{1}{4}\rho^2+3n\right)}\right)^2\rho^2d\psi^2\right) \\ &= \frac{1}{4}\frac{\left(\frac{1}{4}\rho^2+n\right)\left(\frac{1}{4}\rho^2+3n\right)}{\frac{1}{4}\rho^2+\frac{3n}{2}}\left(d\rho^2 \;+\; \left(1+O(\rho^2)\right)\frac{1}{4}\rho^2d\psi^2\right). \end{aligned} \tag{26}$$

Clearly this metric on $\mathbb{R}^2$ is smooth at the origin provided $\frac{1}{2}\psi \in [0,2\pi)$, which is the standard parameterization of $\mathbb{S}^3$. Level sets $r = r_0 > 2n$ are copies of $\mathbb{S}^3$.

Topologically Taub-bolt is a metric on the total space of the tautological bundle over $\mathbb{C}P^1$, which is $\mathcal{O}(-1)$, the same topological space the Burns metric inhabits. Taub-Bolt is not Kähler.

Curvture and asymptotics

This space is Ricci-flat, $|\,\mathrm{Rm}\,|$ decays like $dist^{-3}$, and volumes of balls grow cubically, just as in the Taub-NUT case. The Weyl curvatures are

$$\begin{aligned} W^+ &= \frac{n}{2(r-n)^3}\left(\omega\otimes\omega \;-\; 2Id_{\bigwedge^+}\right) \\ W^- &= \frac{9n}{2(r+n)^3}\left(\omega'\otimes\omega' \;-\; 2Id_{\bigwedge^-}\right). \end{aligned} \tag{27}$$

3.6 Iwai-Katayama generalized Taub-NUTs

Iwai-Katayama [4] termed the following metrics "generalized Taub-NUT" metrics:

$$g \;=\; f(r)\left(dr^2 \,+\, r^2(2\eta_X)^2 + r^2(2\eta_Y)^2\right) \,+\, g(r)(2\eta_Z)^2, \tag{28}$$

although they do not fit the usual rubric in (1). Notice the curious factor of 2 on each coframe, which certainly influences the metric; for instance the standard flat metric is given by $f(r) = \frac{1}{4r}$, $g(r) = \frac{r}{4}$. The standard Taub-NUT of mass $2m$ is $f(r) = 1 + \frac{4m}{r}$, $g(r) = \frac{4m^2}{1+\frac{4m}{r}}$. Such metrics with

$$f(r) \;=\; \frac{a+br}{r}, \qquad g(t) = \frac{r(a+br)}{1+cr+dr^2} \tag{29}$$

were found in [4] to have good dynamical properties—they preserve a Runge-Lenz vector—and so are called "Iwai-Katayama Taub-NUT metrics." For generic values of a, b, c, d they are not too special from a purely Riemannian viewpoint: they are generally non-Kähler and have no special curvature properties, except for cubic curvature decay and that the $W^{\pm}$ always have two eigenvalues each. The exception is when $d = \frac{1}{4}c^2$, in which case these metrics are half-conformally flat.

Curvature and asymptotics

These are scalar flat if and only if $c = \frac{2b}{a}$ and $d = \frac{b^2}{a^2}$, which is the standard Taub-NUT metric, and half-conformally flat if and only if $d = \frac{1}{4}c^2$, in which case

$$\begin{aligned} W^+ = 0, \qquad W^- \;&=\; \left(\frac{c}{(2+cr)^2(a+br)}\right)\left(3\omega'\otimes\omega' - 2Id_{\bigwedge^-}\right) \\ R \;&=\; \frac{3}{2}\frac{(ac-2b)(8a+(6b+ac)r)}{(2+cr)^2(a+br)^3} \end{aligned} \tag{30}$$

The Ricci tensor is too complicated to write down usefully, but also decays like $dist^{-3}$. These violate Derdzinski's theorem in either orientation (described in §2.2 or [1]), so clearly they are not Kähler except in the scalar flat case. Weyl decays like $dist^{-3}$, and scalar curvature decays like $dist^{-4}$. Balls grow like $O(r^3)$, so these manifolds are ALF. Asymptotically these metrics resemble the Taub-NUT metric, since the Hopf fiber has bounded length and the perpendicular distribution grows linearly.

3.7 Miyake generalized Taub-NUTs

Following on the Iwai-Katayama metrics, Miyake [8] built a 2-parameter family of anti-deSitter style Taub-NUT metrics:

$$g \;=\; \frac{-24bc\left(c+2br+r^2\right)}{r\left(bc+2cr+br^2\right)^2)}\left(dr^2+4r^2\eta_X^2+4r^2\eta_Y^2\;+\;4\left(\frac{r(c-r^2)}{(c+2br+r^2)}\right)^2\eta_Z^2\right) \tag{31}$$

The parameters b, c either satisfy $c>0$ and $-\sqrt{c}\leq b<0$, or else both c, b are negative. However if the metric's sign is changed then positive values of b and c are allowed, although the Einstein constant changes, and removal of coordinate singularities becomes a delicate question. The coordinate r ranges within the interval $\left(0,\,\frac{c}{b}\left(-1+\sqrt{1-\frac{c^2}{b}}\right)\right)$. At 0 these metrics can be completed by adjoining a nut (a single point), and they are complete as $r\to\frac{c}{b}\left(-1+\sqrt{1-\frac{c^2}{b}}\right)$.

These metrics are Einstein and half conformally flat. Within the allowed parameter range, they are never Kähler. The single case where they are Kähler, with reversed orientation and the sign on the metric changed, is $b=c=+1$; this metric has scalar curvature $+1$, and after studying the coordinate singularities, one verifies that it is actually a multiple of the Fubini-Study metric on $\mathbb{C}P^2$.

Curvature and asymptotics

Scalar curvature is -1 for all allowed values of b, c, and all metrics are Einstein: $\mathring{\mathrm{Ric}}=0$. The Weyl curvatures are

$$W^+\;=\;0,\qquad W^-\;=\;-\frac{(bc+2cr+r^2)^3}{24bc(c+2br+r^2)^3}\left(3\omega'\otimes\omega'\;-\;2Id_{\bigwedge^-}\right) \tag{32}$$

Clearly none of the curvatures decay, except W^+ which is already zero. Ball volumes expand exponentially.

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